ADVENTURES WITH TRIANGLES IN BILLIARDS

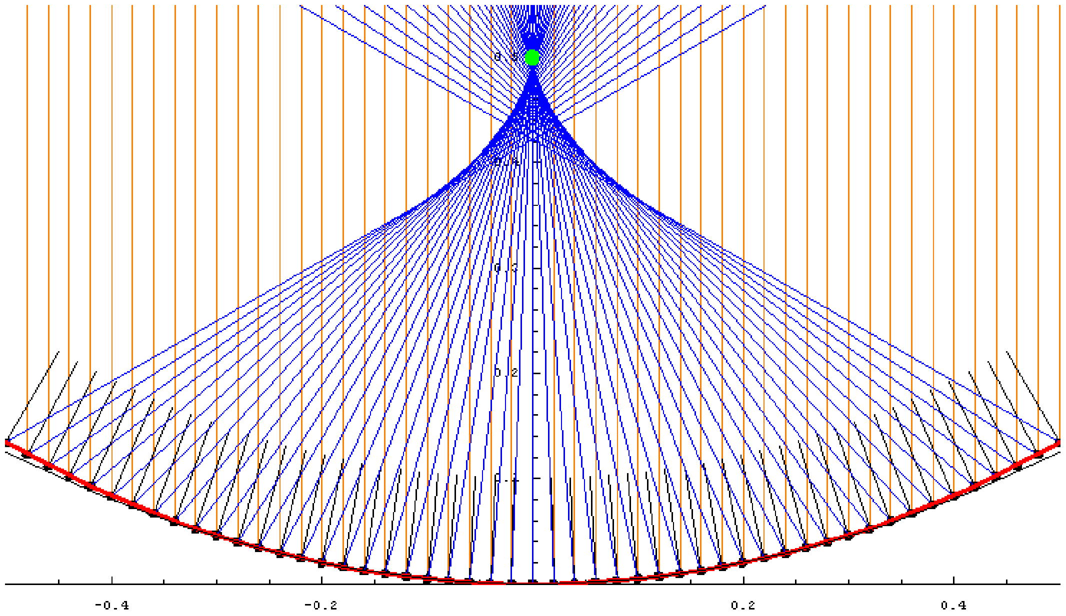
DAN S. REZNIK

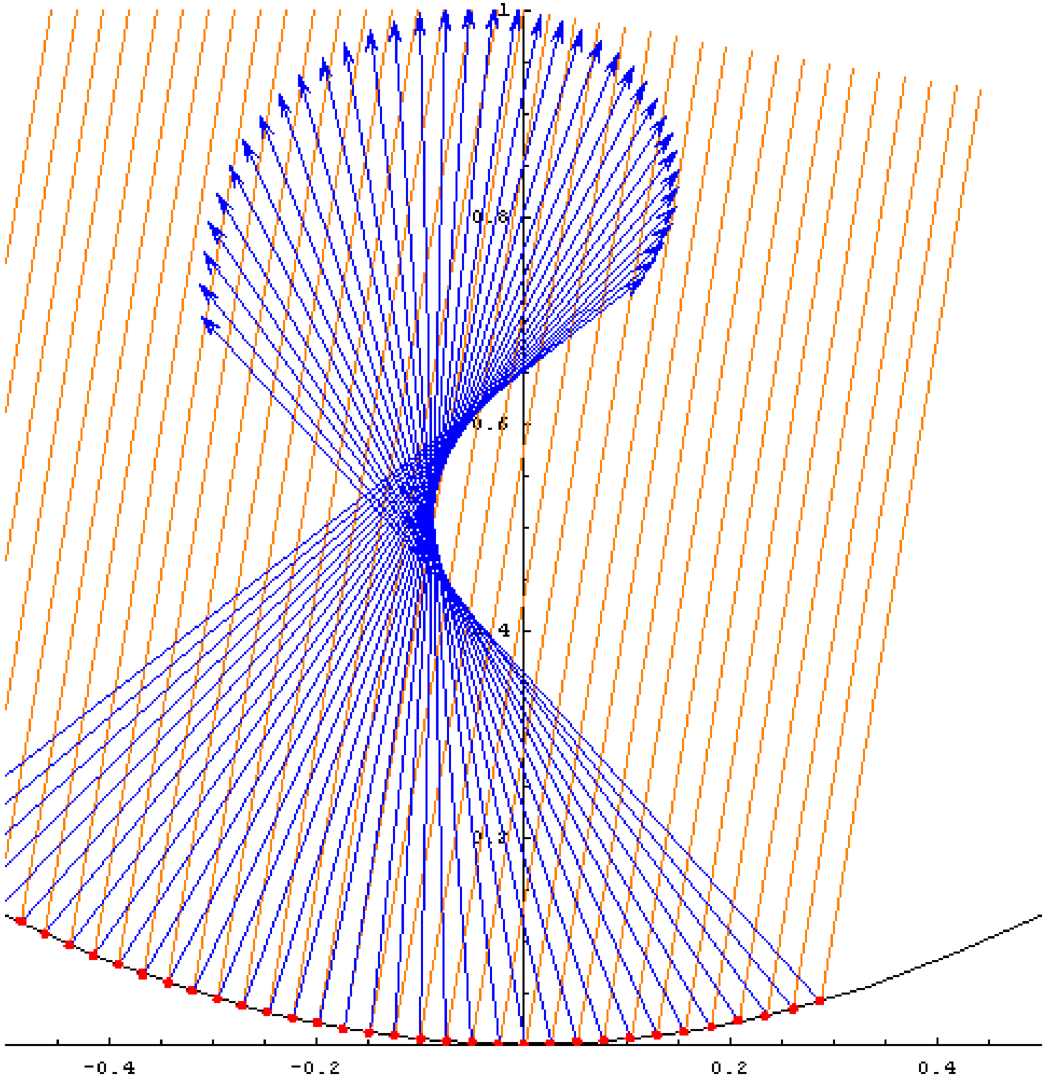
RONALDO GARCIA

JAIR KOILLER

IMPA, RIO DE JANEIRO, JULY 29TH, 2019

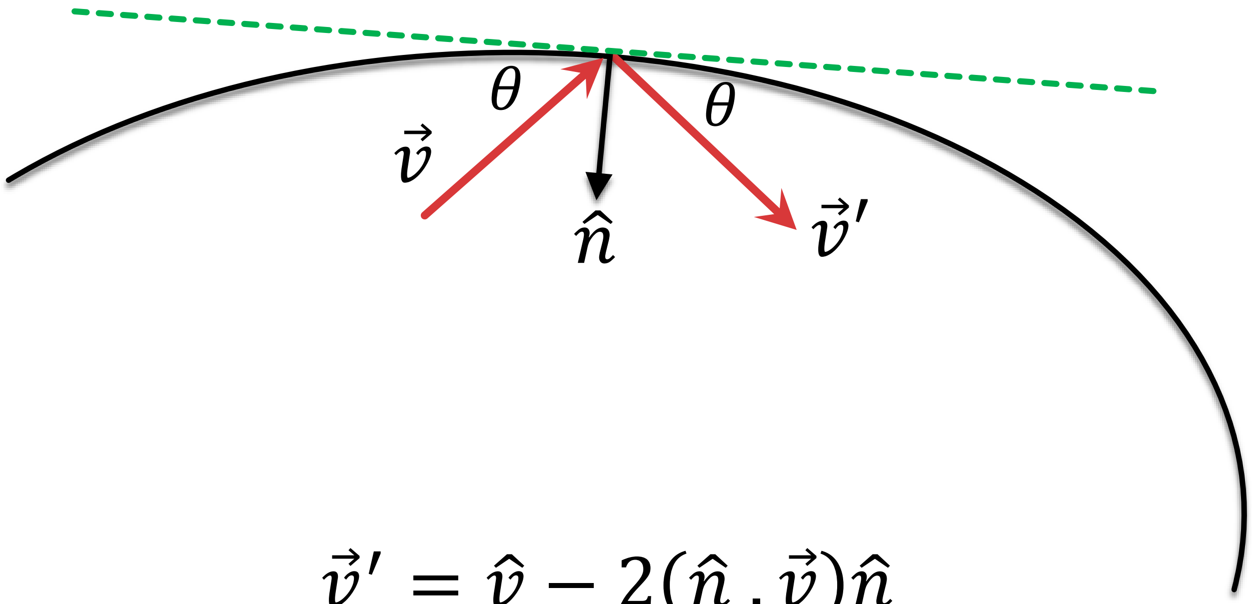
RAY TRACING CURVED MIRRORS 2006



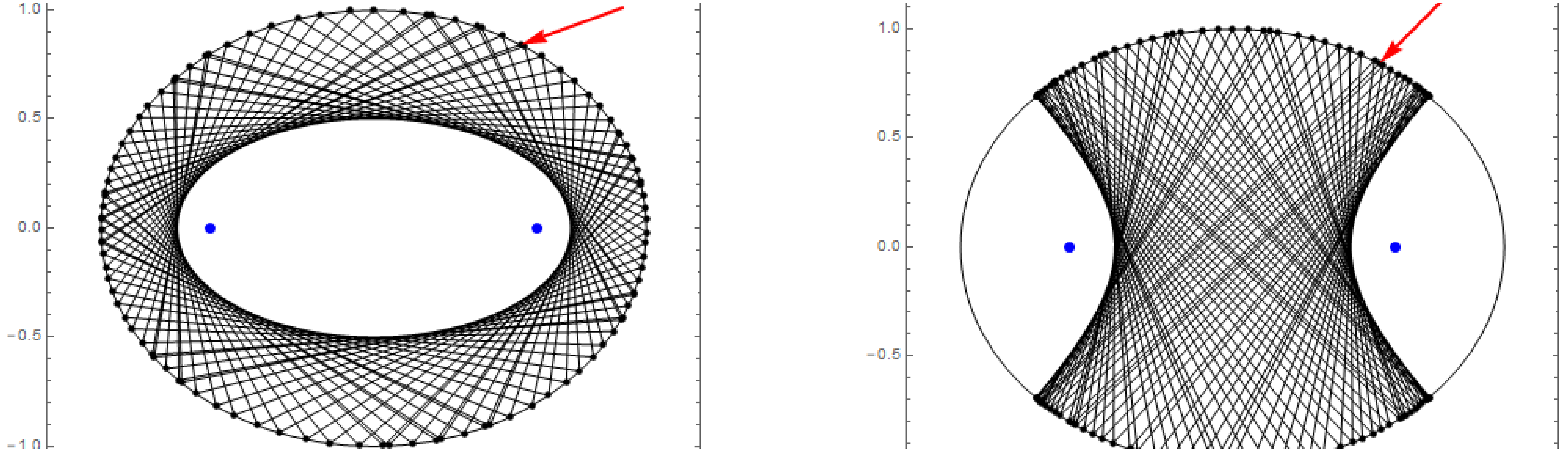


POOL TABLE / JEU DE BILLIARDS

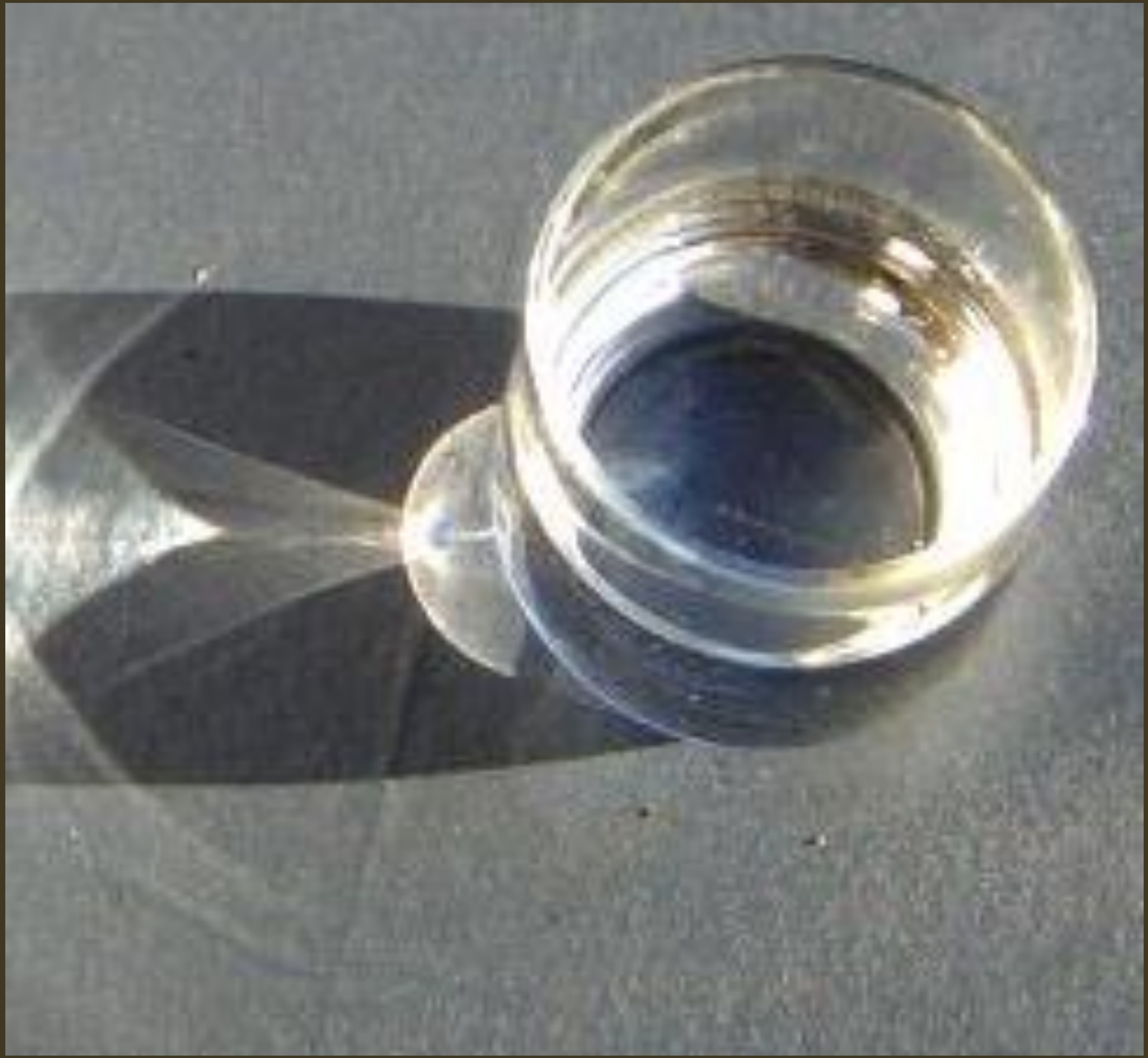
ELASTIC COLLISION



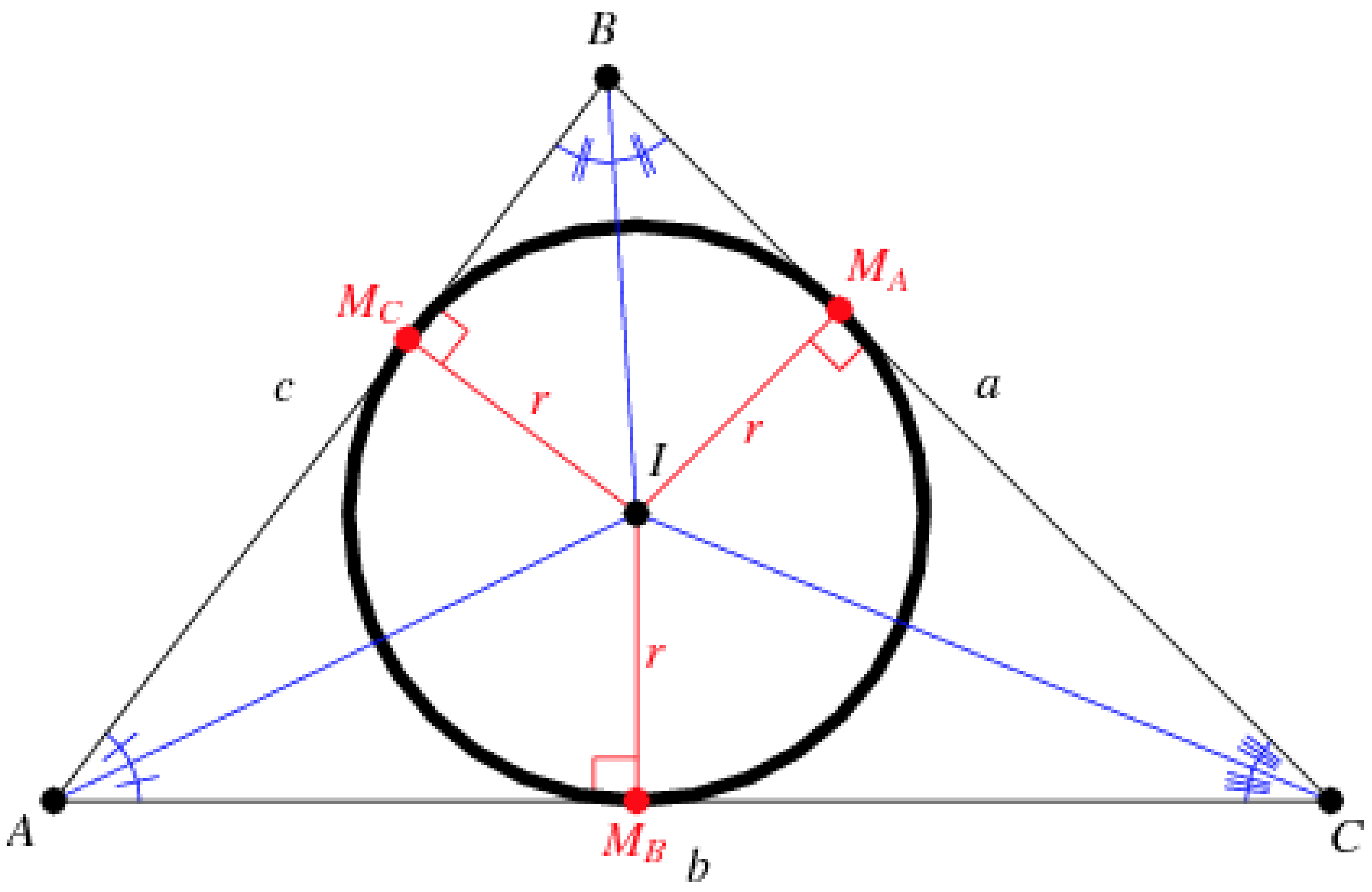
$$\vec{v}' = \hat{v} - 2(\hat{n}\,.\,\vec{v})\hat{n}$$



CAUSTICS
VIDEO



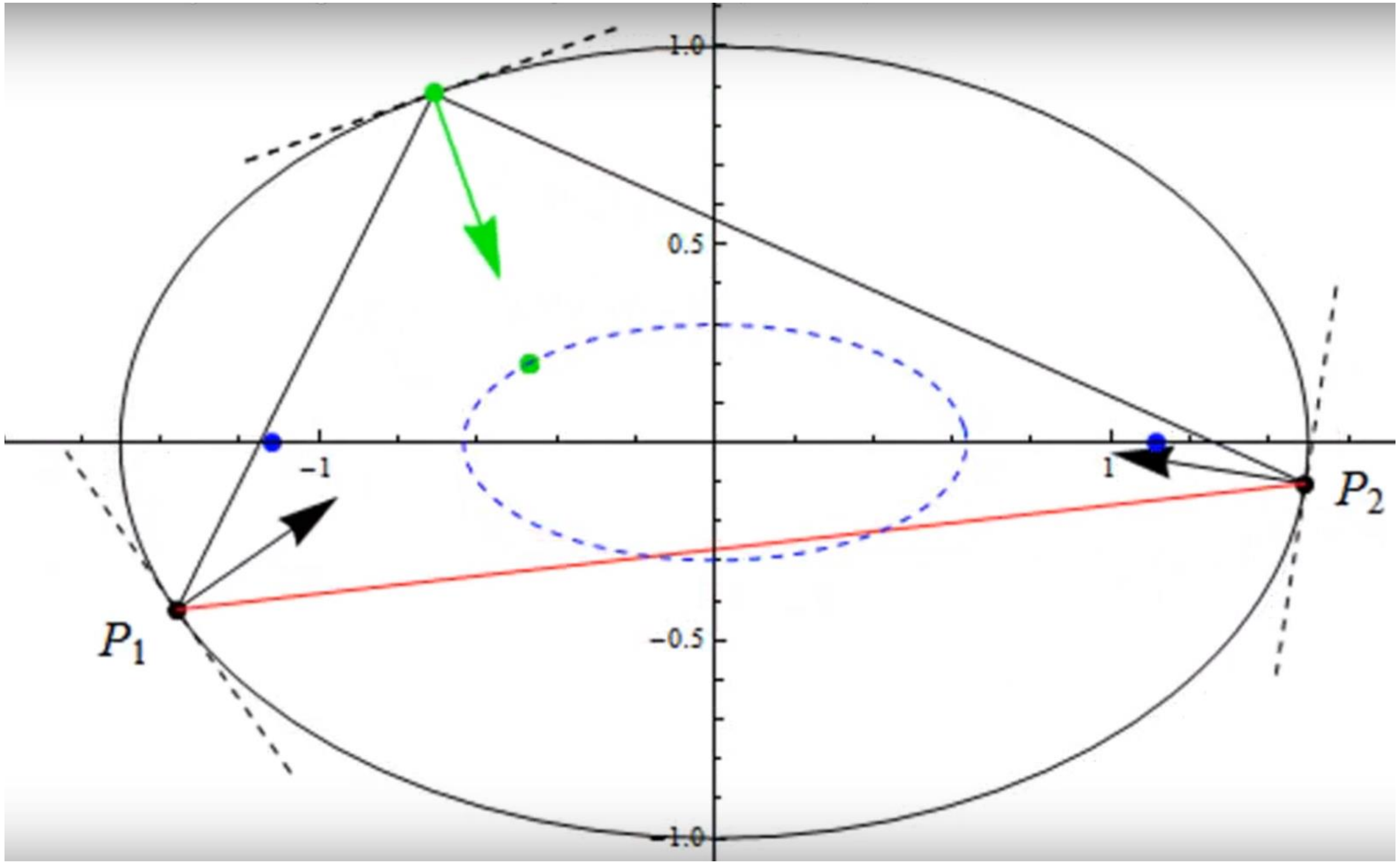
INCENTER



TRIANGULAR ORBITS (2011)

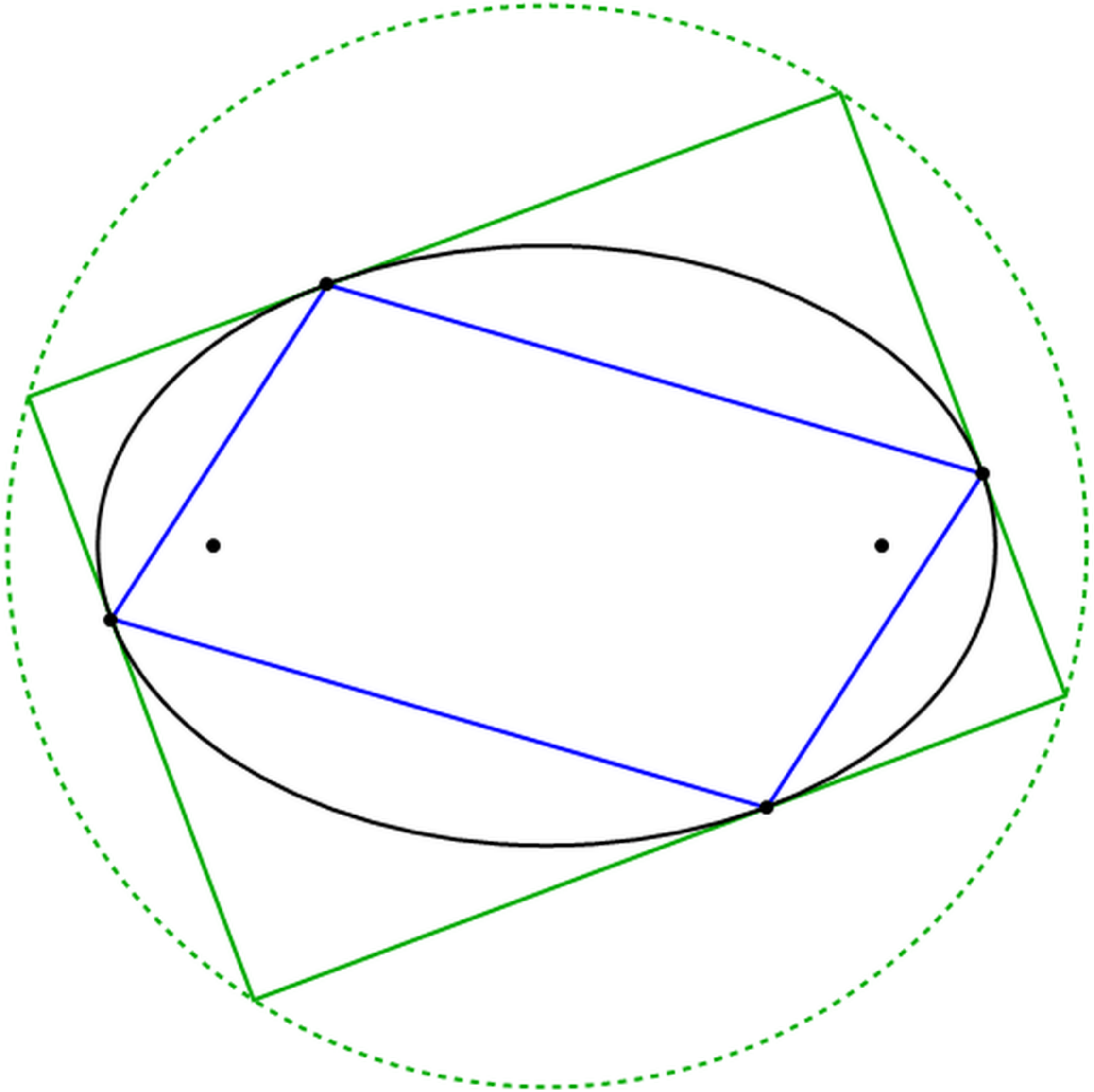
VIDEO 1

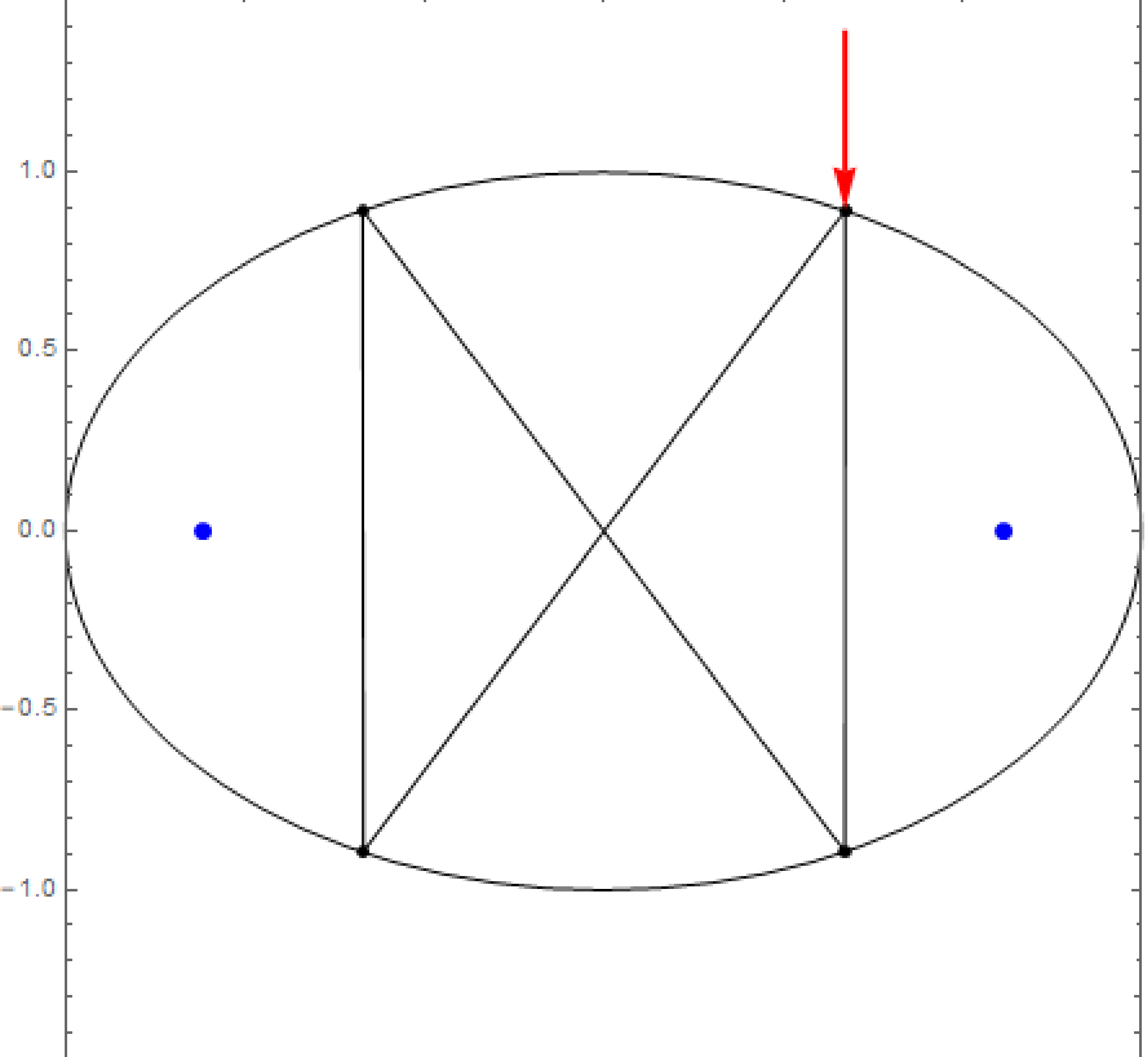
VIDEO 2



MONGE'S ORTHOPTIC CIRCLE

VIDEO





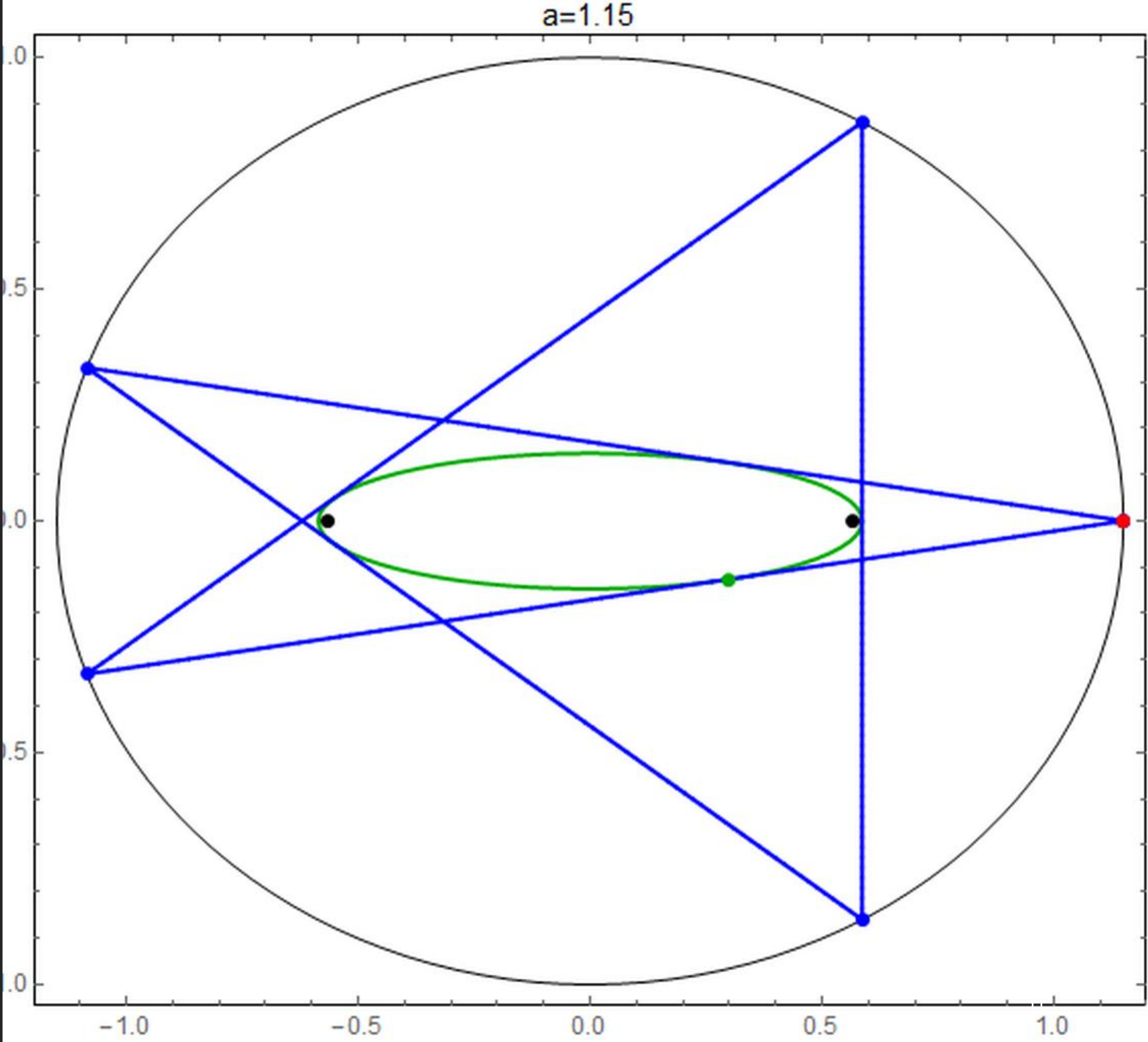
N=4 SELF-INTERSECTING

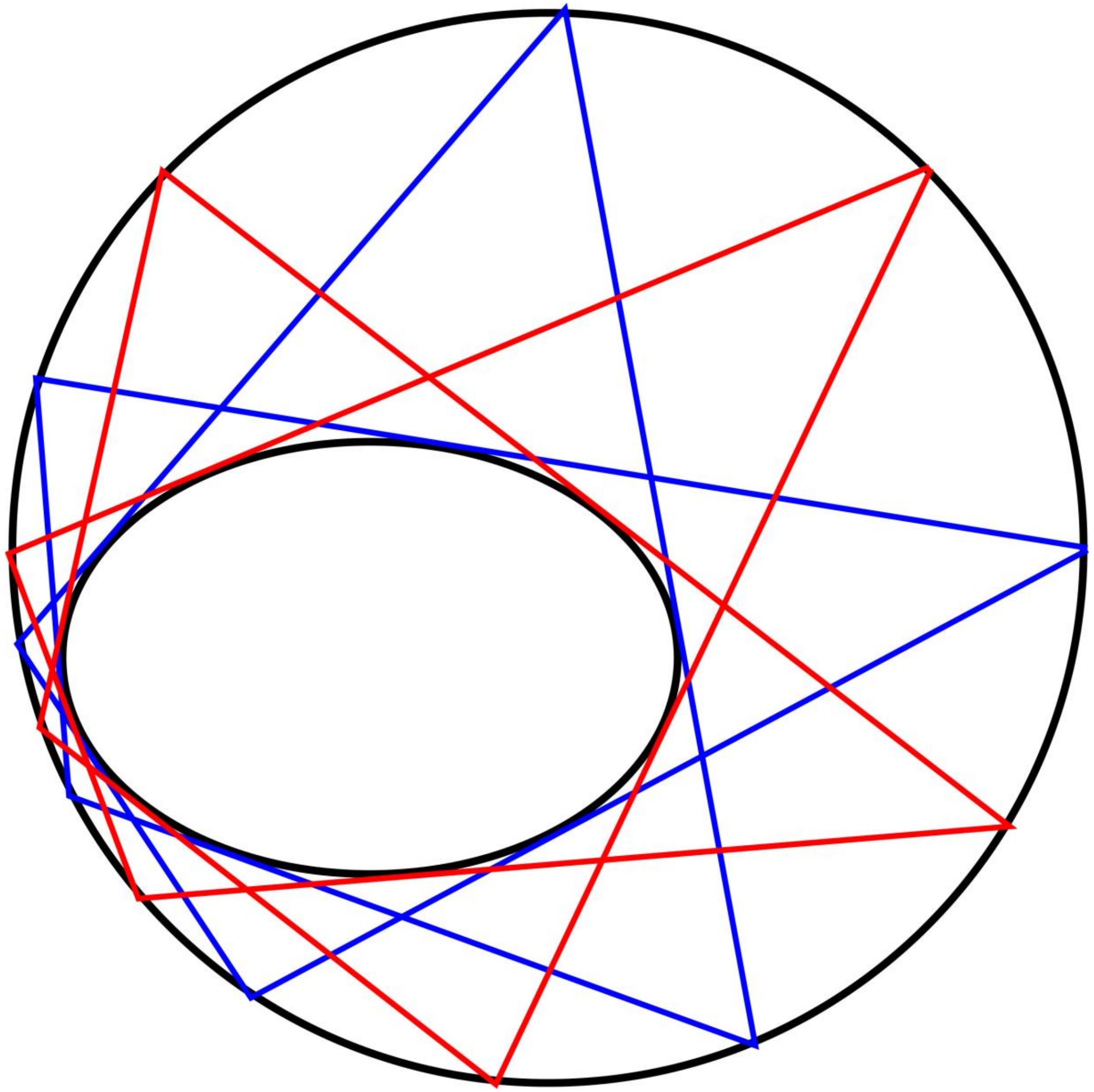
VIDEO

SELF-INTERSECTING PENTAGRAM

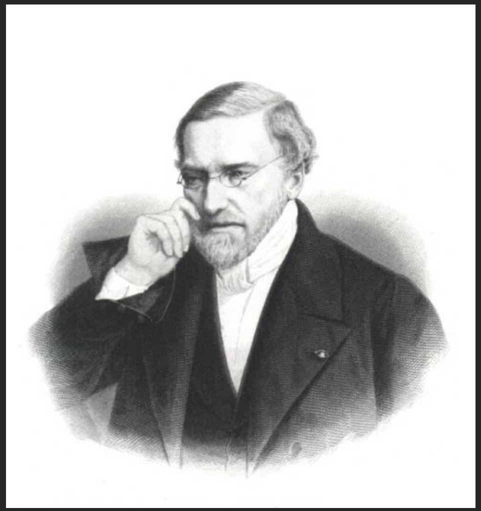
[VIDEO]

[APPLET]



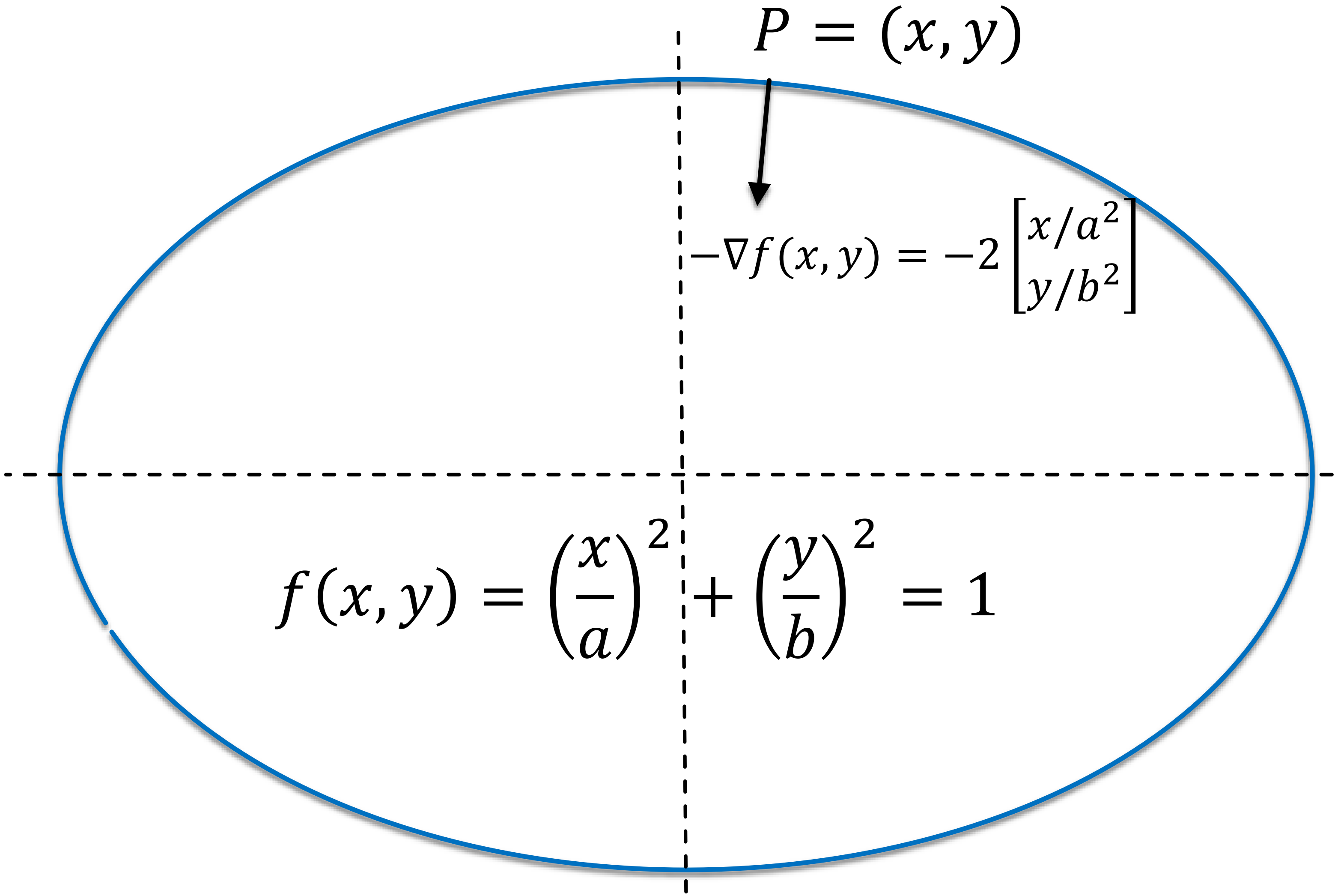


PONCELET “PORISM”



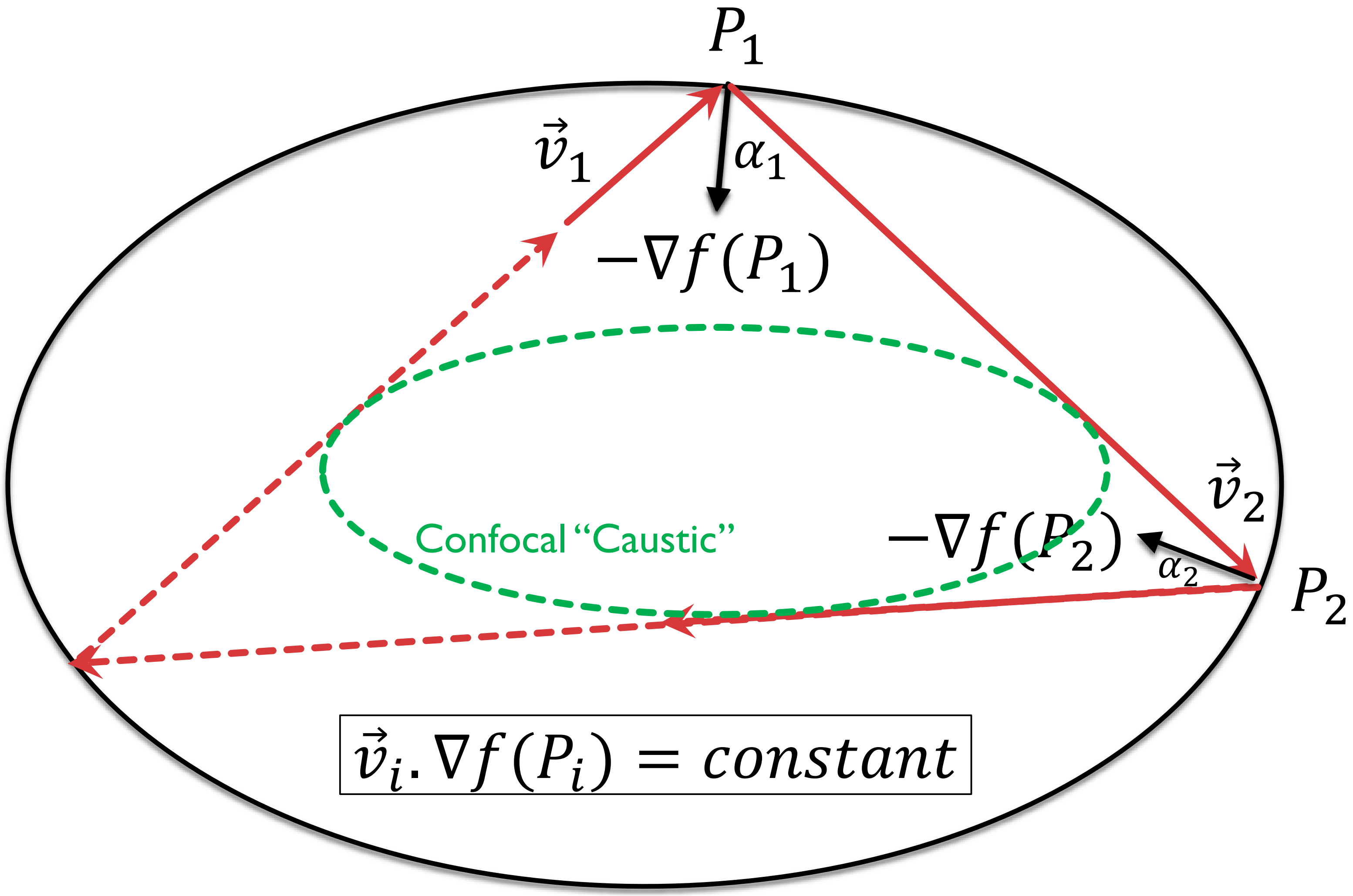
Jean-Victor Poncelet
1788–1867

ELLIPSE

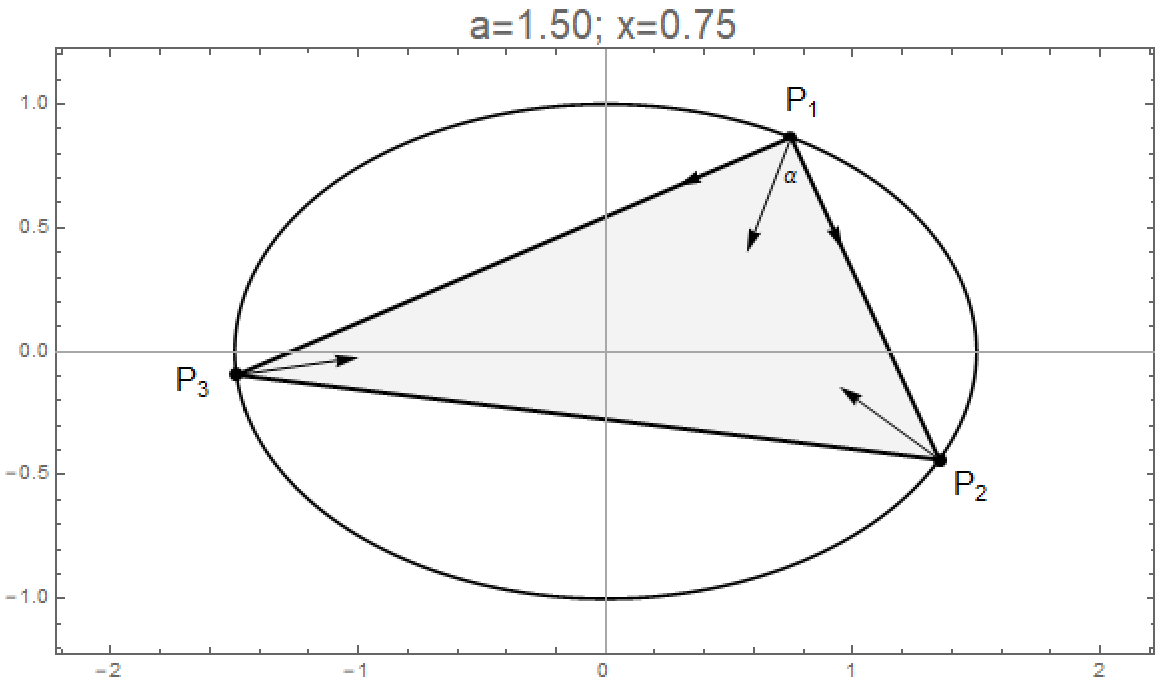

$P = (x, y)$
$-\nabla f(x,y) = -2\begin{bmatrix} x/a^2 \\ y/b^2 \end{bmatrix}$
$f(x,y) = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

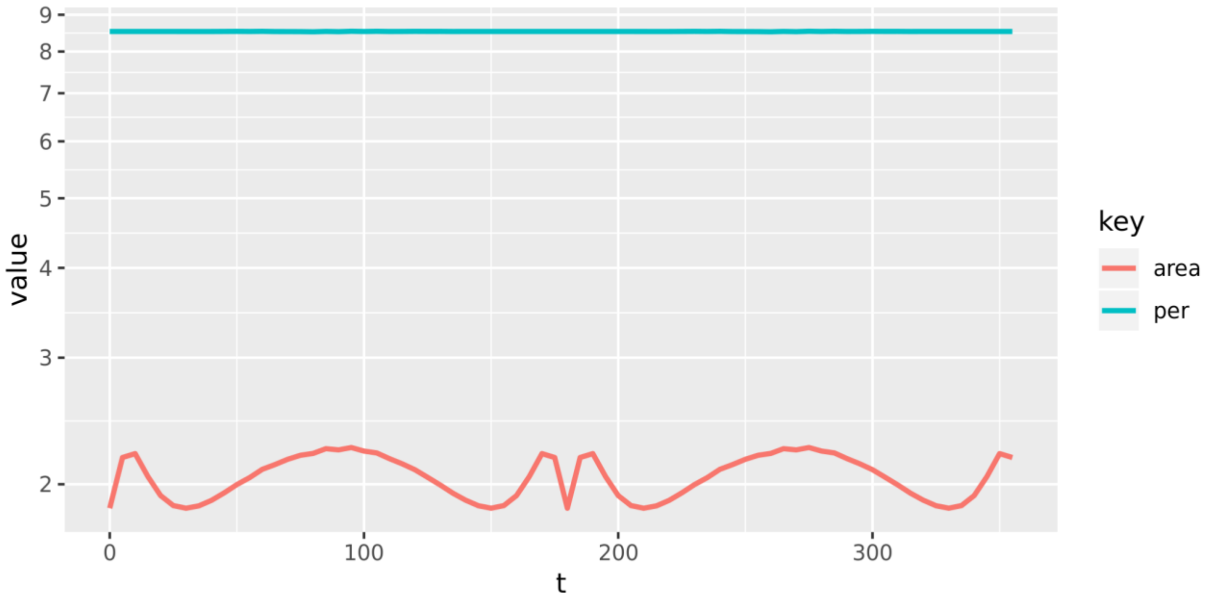
Ferdinand Joachimsthal
1818-1861

CONSERVATION OF MOMENTUM



PERIMETER IS MAXIMAL & CONSTANT [INTEGRABLE HAMILTONIAN]





OUTLINE

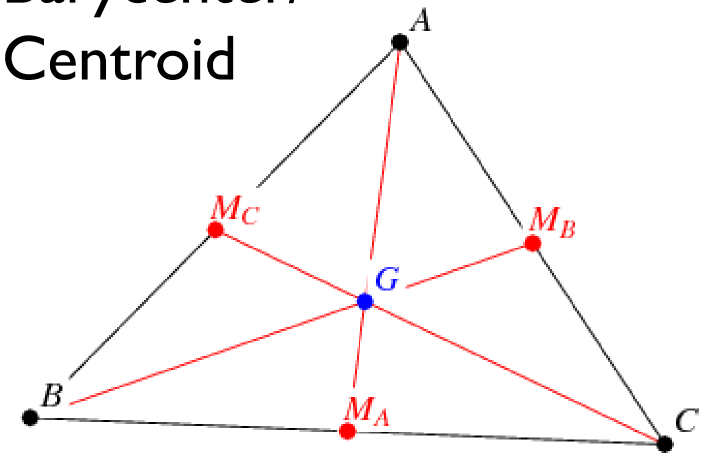
- Loci of Triangular Centers
- Mittenpunkt is Stationary
- Two Constants of Motion
- Experimental Setup
- Conclusion
- Videos / Questions

ELLIPTIC LOCI

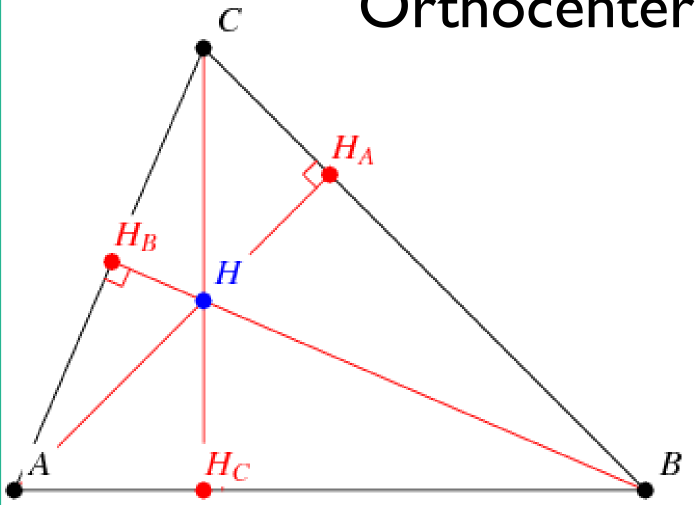
OF TRIANGULAR CENTERS

MAJOR TRIANGULAR CENTERS

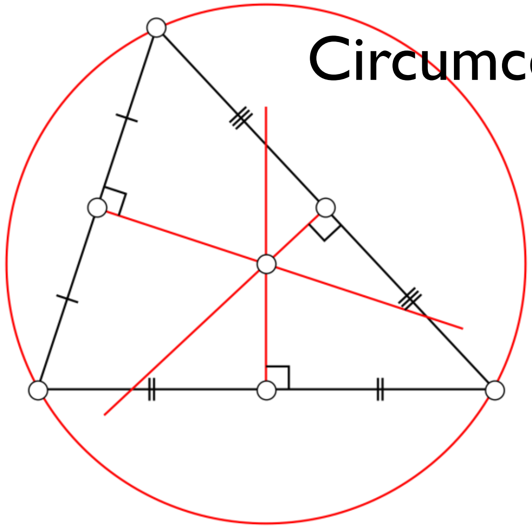
Barycenter/ Centroid



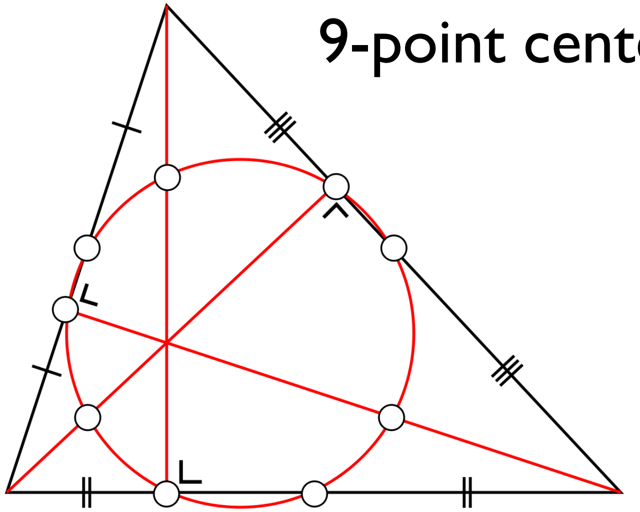
Orthocenter



Circumcenter



9-point center



ON THE INCENTERS OF TRIANGULAR ORBITS IN ELLIPTIC BILLIARD

by Olga ROMASKEVICH *)

ABSTRACT. We consider 3-periodic orbits in an elliptic billiard. Numerical experiments conducted by Dan Reznik have shown that the locus of the centers of inscribed circles of the corresponding triangles is an ellipse. We prove this fact by the complexification of the problem coupled with the complex law of reflection.

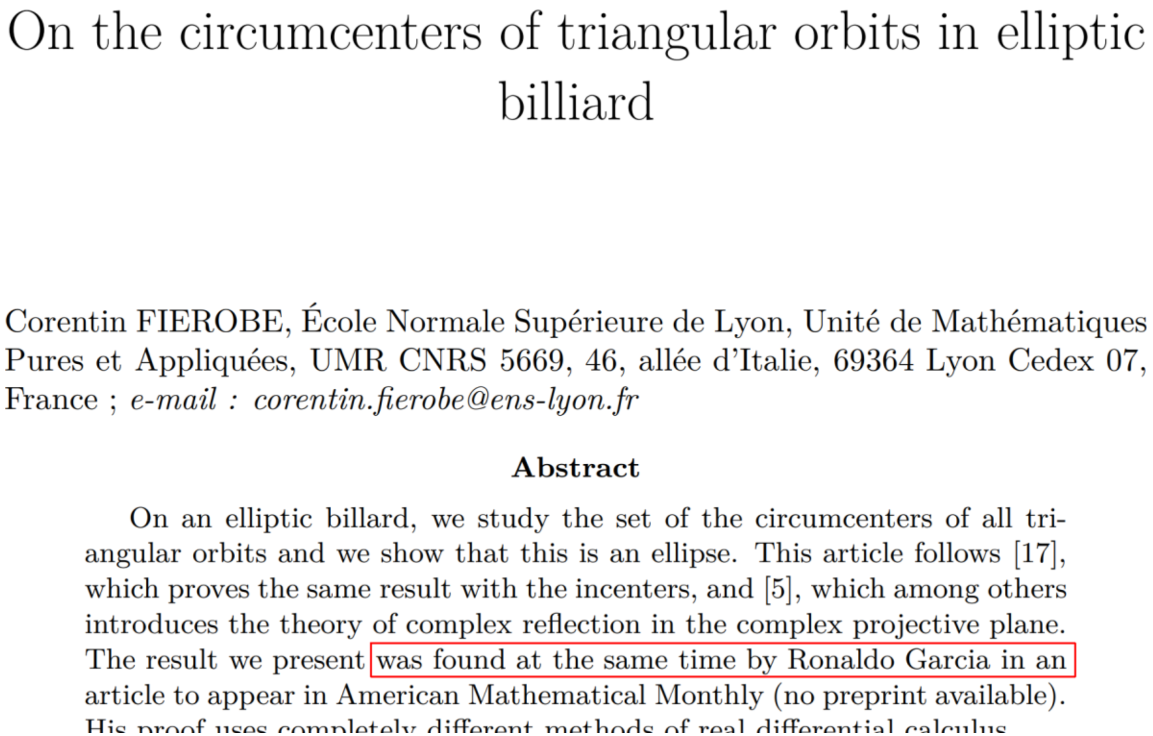
LOCUS OF **INCENTER** IS ELLIPTIC, 2013
PROF. OLGA ROMASKEVICH,
UNIVERSITÉ DE RENNES, FRANCE
PDF

CENTERS OF INSCRIBED CIRCLES IN TRIANGULAR ORBITS OF AN ELLIPTIC BILLIARD

RONALDO A. GARCIA*

ABSTRACT. The locus of centers of inscribed circles in triangles, the 3-periodic orbits of an elliptic billiard, is also an ellipse. In this work we obtain the canonical equation of this ellipse, complementing the previous results obtained by O. Romaskevich in [7].

LOCUS OF **CIRCUMCENTER** IS ELLIPTIC, 2016 (I)
PROF. RONALDO GARCIA
UNIVERSIDADE DE GOIÁS, BRAZIL
PDF



On the circumcenters of triangular orbits in elliptic billiard

Corentin FIEROBE, École Normale Supérieure de Lyon, Unité de Mathématiques Pures et Appliquées, UMR CNRS 5669, 46, allée d'Italie, 69364 Lyon Cedex 07, France ; *e-mail : corentin.fierobe@ens-lyon.fr*

Abstract

On an elliptic billard, we study the set of the circumcenters of all triangular orbits and we show that this is an ellipse. This article follows [17], which proves the same result with the incenters, and [5], which among others introduces the theory of complex reflection in the complex projective plane. The result we present was found at the same time by Ronaldo Garcia in an article to appear in American Mathematical Monthly (no preprint available). His proof uses completely different methods of real differential calculus

LOCUS OF **CIRCUMCENTER** IS ELLIPTIC, 2019 (II)
CORENTIN FIEROBE,
ENS-LYON, FRANCE
PDF

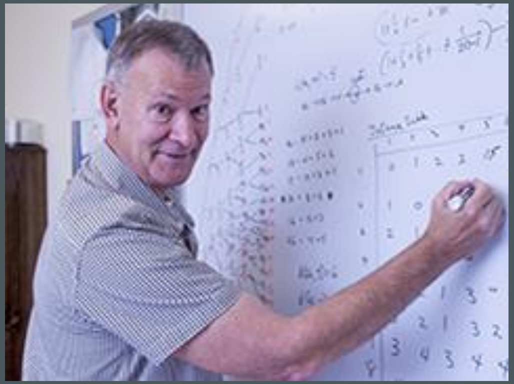
Centers of mass of Poncelet polygons, 200 years after

Richard Schwartz* Serge Tabachnikov†

1 A letter from Saratov

During his last trip to Moscow, the second author of this article came into possession of a remarkable mathematical letter. The custodian of the letter, a Russian businessman X who wished to remain anonymous, presented the letter to Tabachnikov at the end of his lecture on configuration theorems in projective geometry [11] (like many successful contemporary Russian entrepreneurs, Mr. X has a degree in mathematics).

LOCUS OF **BARYCENTER** (FOR ALL N) IS ELLIPTIC, 2016
PROFS. RICHARD SCHWARTZ (BROWN) & SERGEI TABACHNIKOV (PENN STATE)
PDF



KIMBERLING'S ENCYCLOPEDIA OF TRIANGULAR CENTERS (ETC)

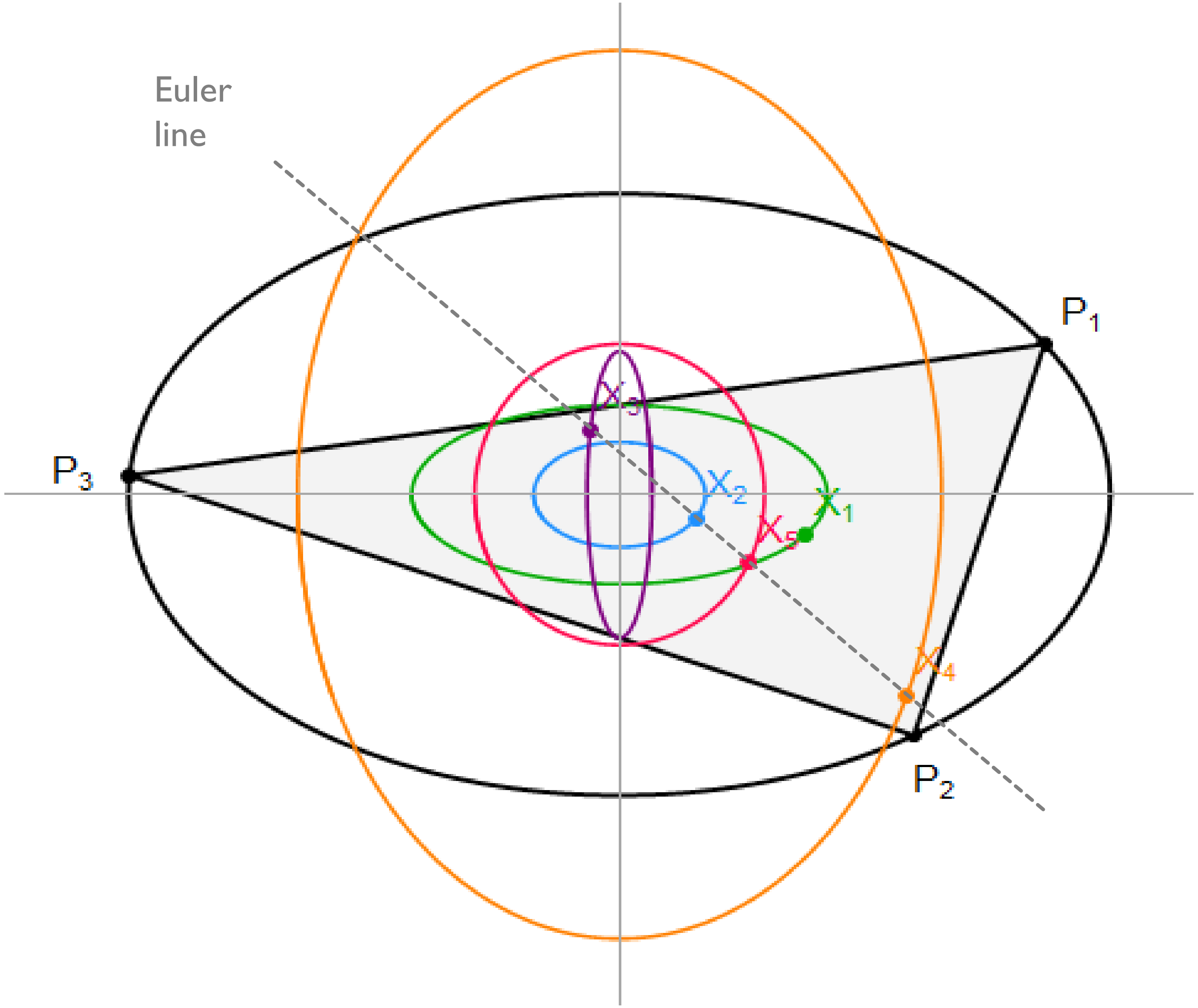
Clark Kimberling's

ENCYCLOPEDIA OF TRIANGLE CENTERS - ETC

Tables Glossary Search Sketches Links Thanks

This is PART 1: Introduction and Centers X(1) - X(1000)

PART 1:	Introduction and Centers X(1) - X(1000)
PART 2:	Centers X(1001) - X(3000)
PART 3:	Centers X(3001) - X(5000)
PART 4:	Centers X(5001) - X(7000)
PART 5:	Centers X(7001) - X(10000)
PART 6:	Centers X(10001) - X(12000)
PART 7:	Centers X(12001) - X(14000)
PART 8:	Centers X(14001) - X(16000)
PART 9:	Centers X(16001) - X(18000)
PART 10:	Centers X(18001) - X(20000)
PART 11:	Centers X(20001) - X(22000)
PART 12:	Centers X(22001) - X(24000)
PART 13:	Centers X(24001) - X(26000)
PART 14:	Centers X(26001) - X(28000)
PART 15:	Centers X(28001) - X(30000)
PART 16:	Centers X(30001) - X(32000)
PART 17:	Centers X(32001) - X(34000)
PART 18:	Centers X(34001) - X(36000)
PART 19:	Centers X(36001) - X(38000)
PART 20:	Centers X(38001) - X(40000)

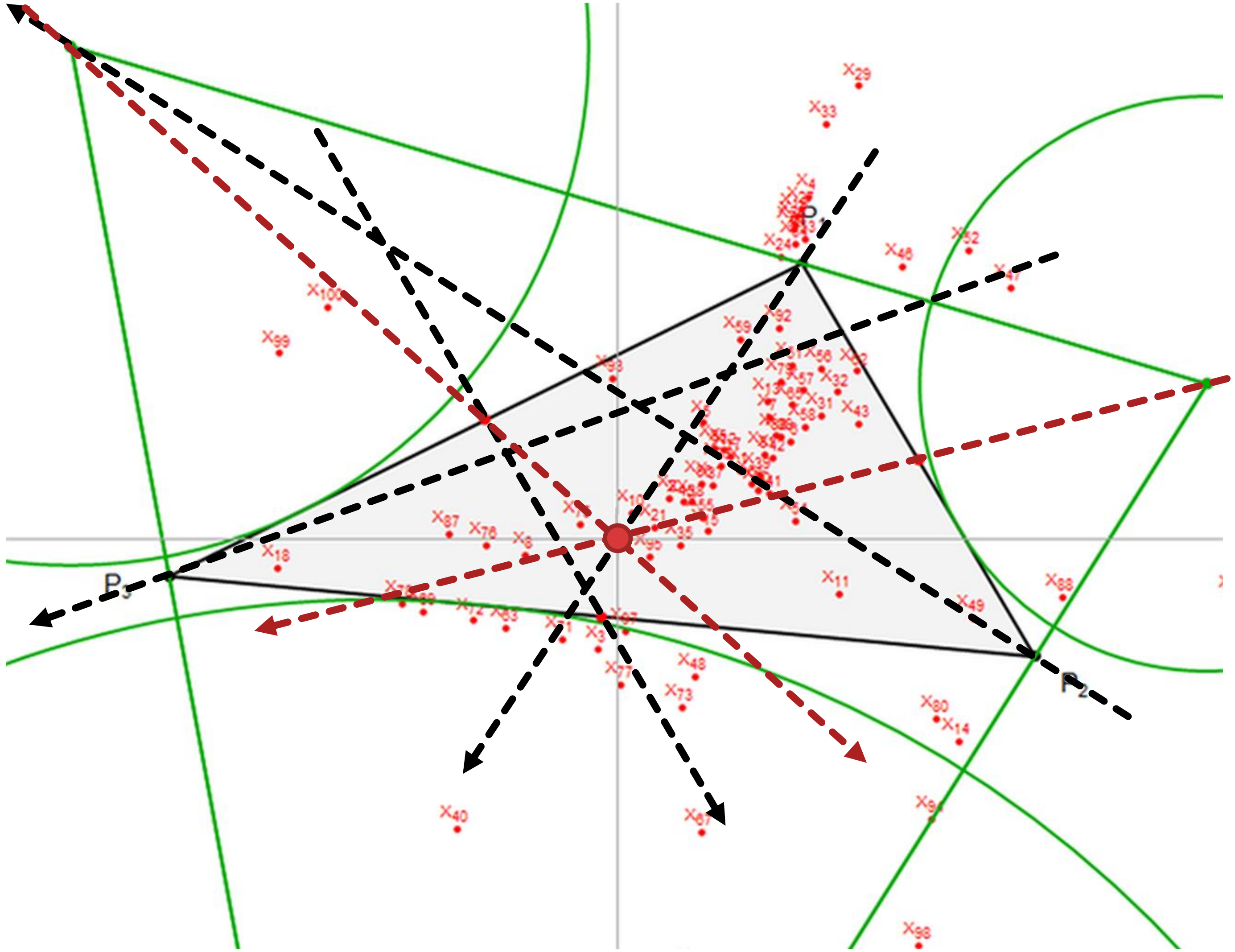


ELLIPTIC LOCI

[VIDEO]

MITTENPUNKT

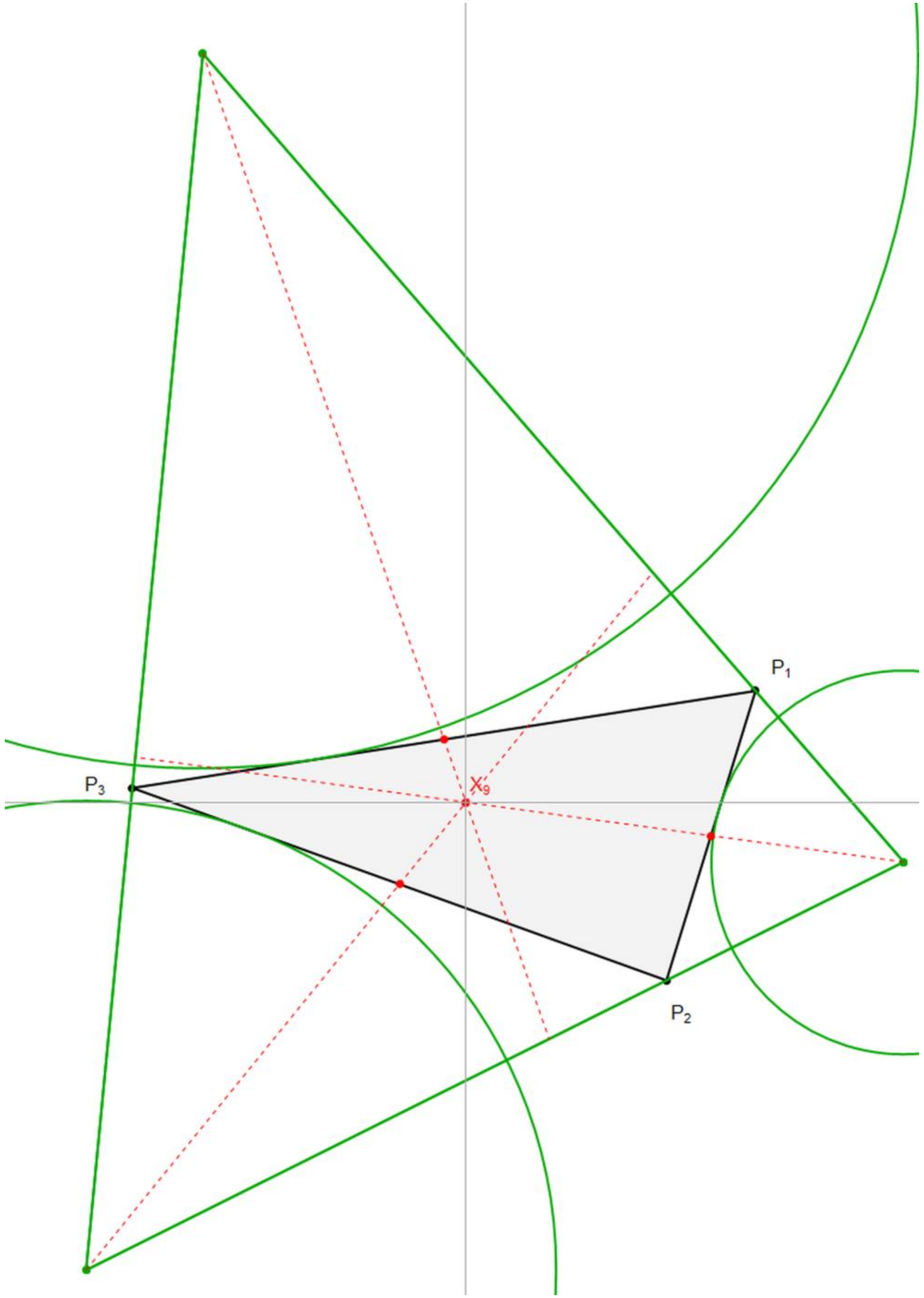
STATIONARITY



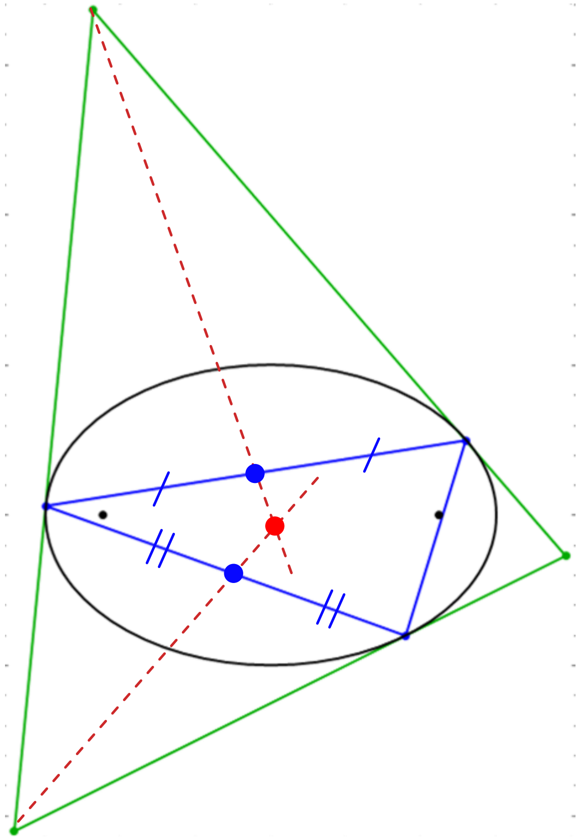
HUNTING FOR A FIXED POINT

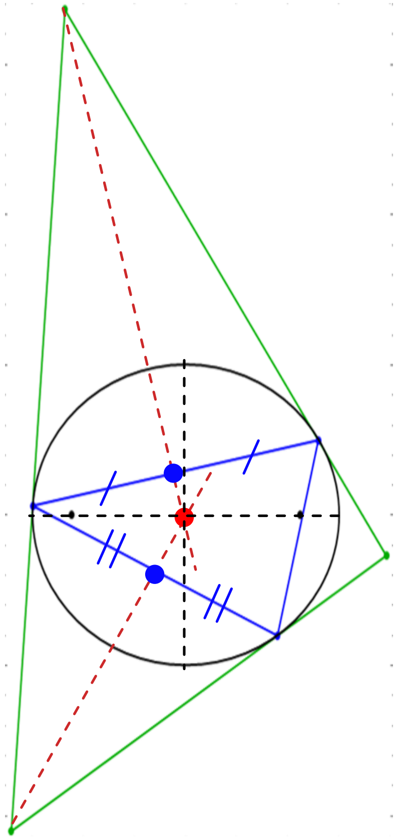
Christian Heinrich von Nagel
(1803–1882)

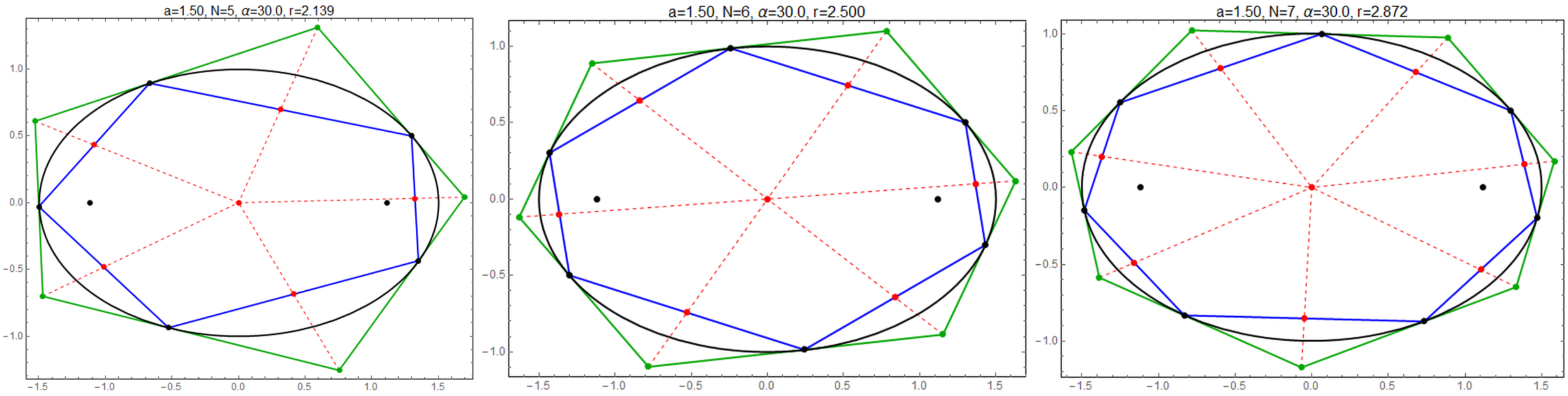
THE MITTENPUNKT VIDEO



PROOF SKETCH
(PROF. OLGA ROMASKEVICH)



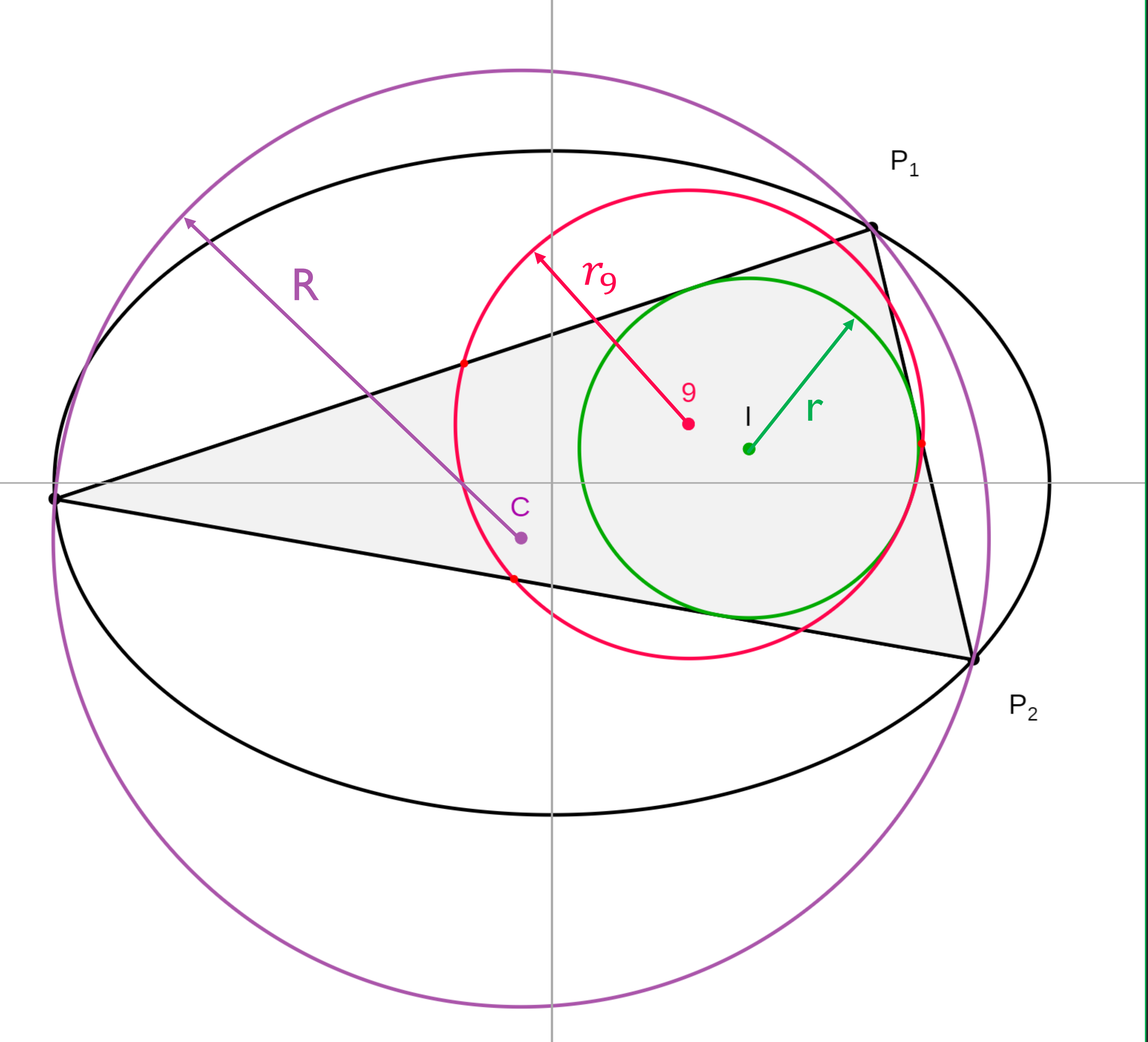




GENERALIZE MITTENPUNKT FOR N>3

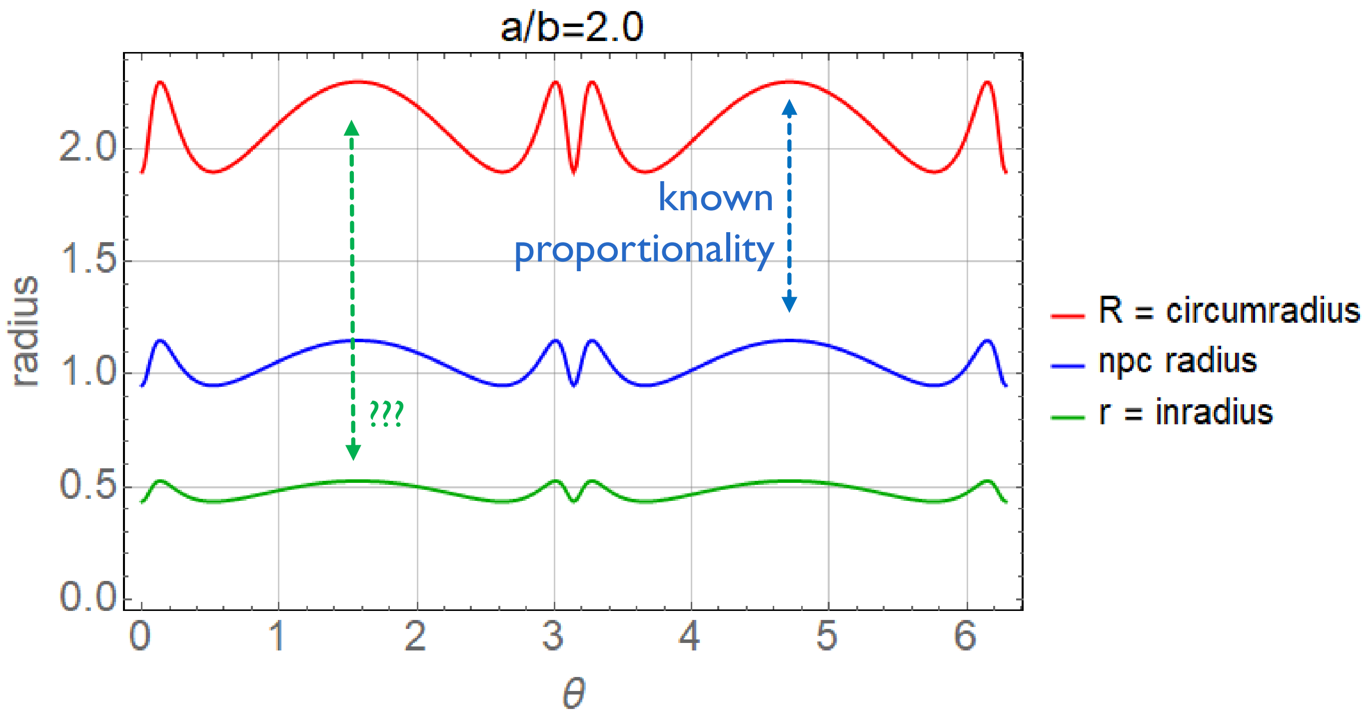
CONSTANTS OF MOTION

* COSINE SUM
* COSINE PRODUCT

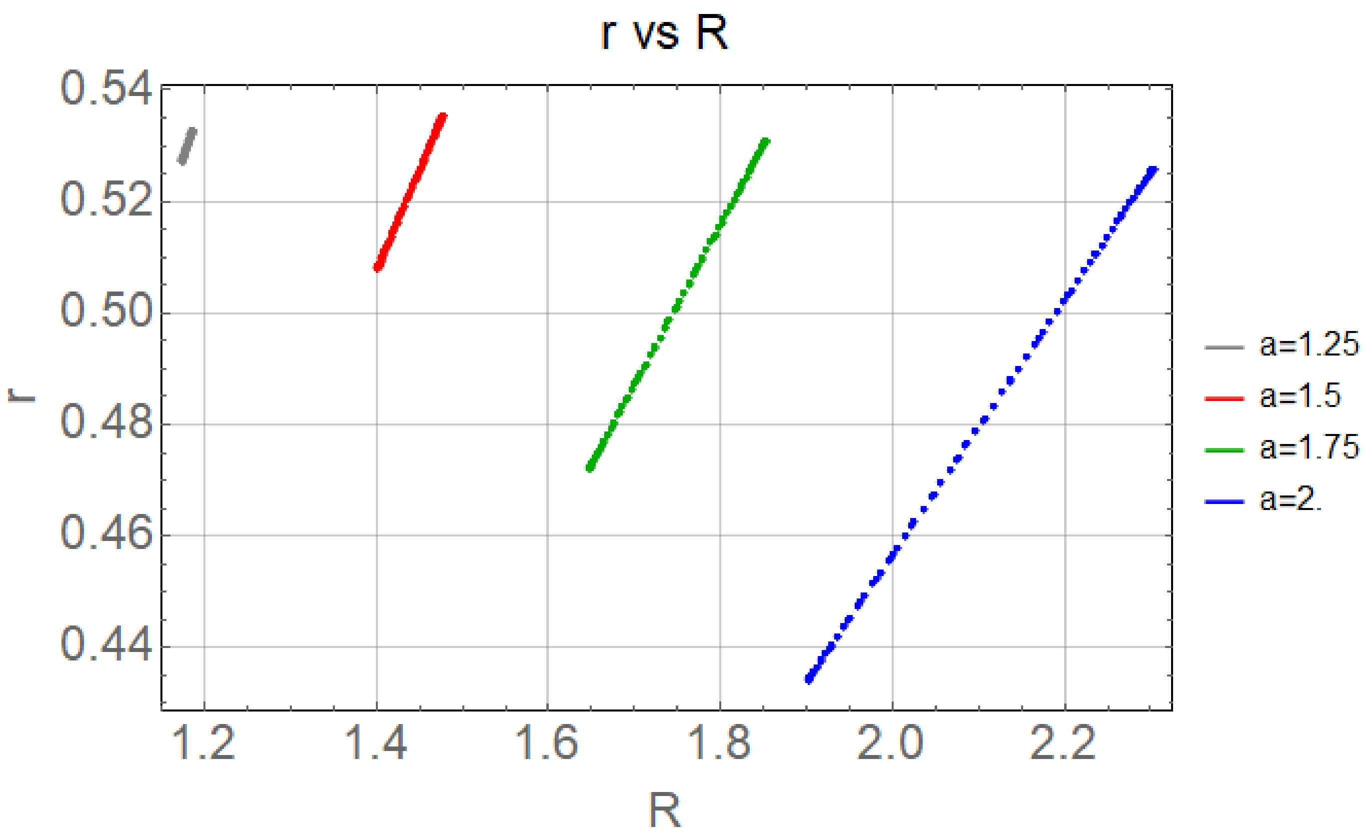


THREE LITTLE CIRCLES

COMPARING RADII

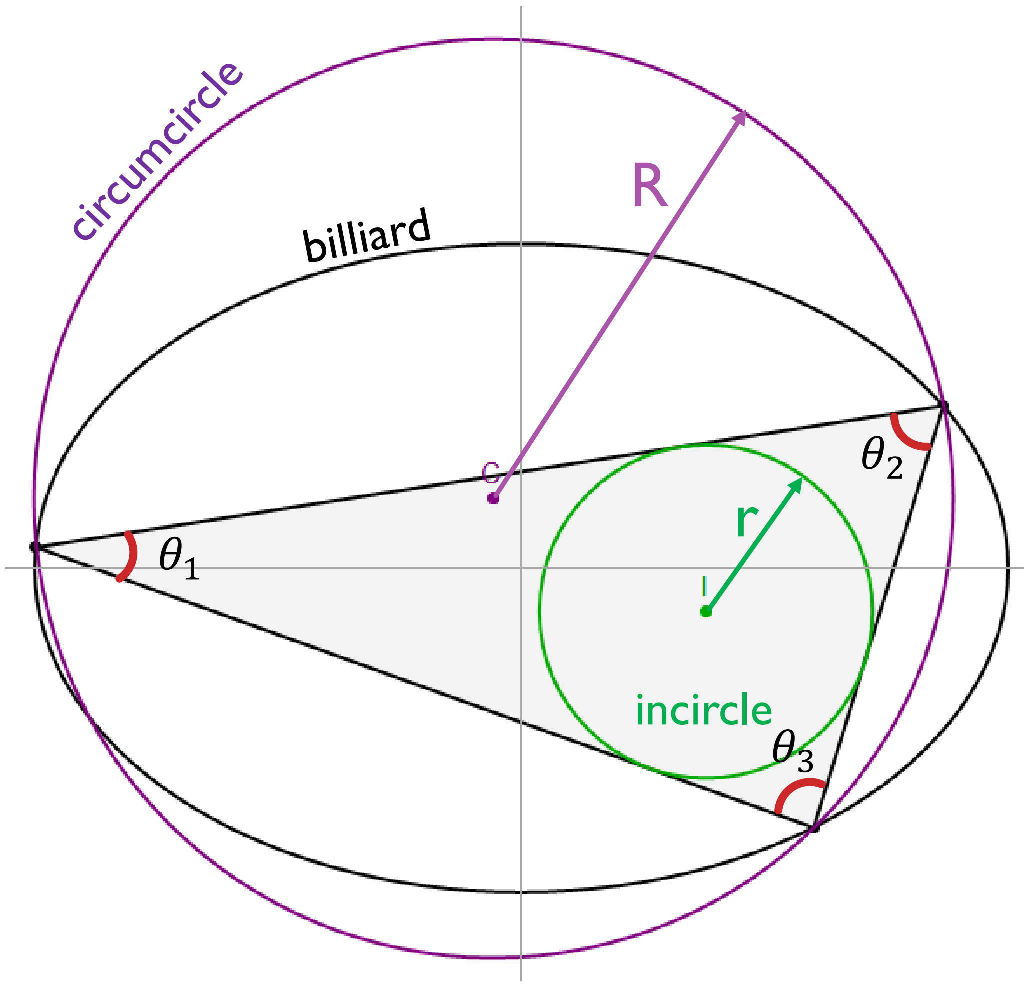


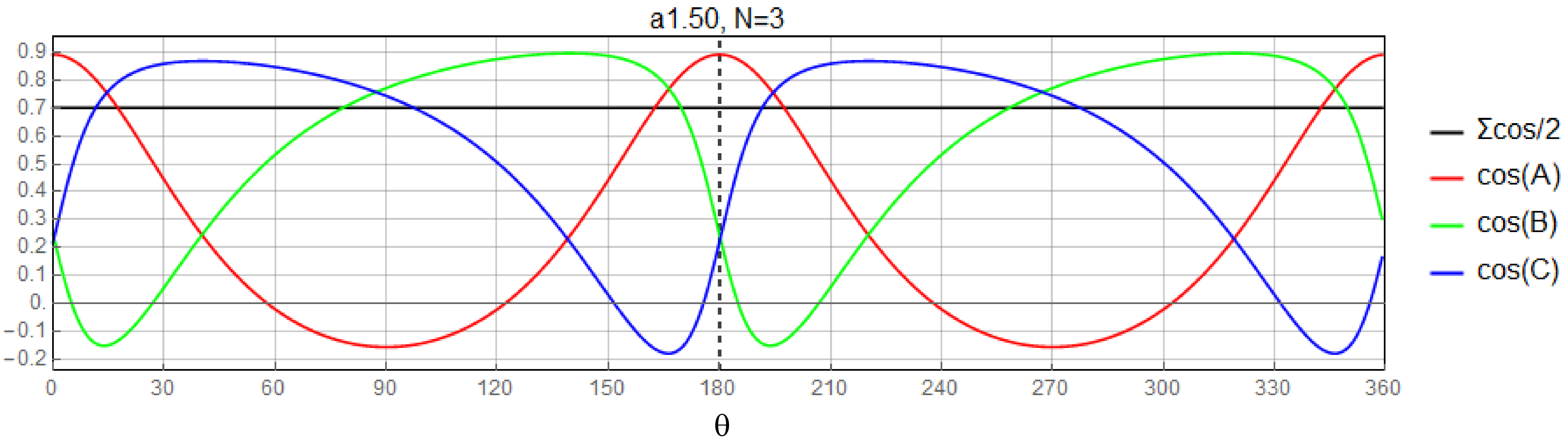
SCATTER INRADIUS VS. CIRCUMRADIUS



WONDER IDENTITY

$$\frac{r}{R} = \sum_{i=1}^{3} \theta_i$$





SUM OF COSINES = CONSTANT

MAY 6^{TH}, 2019

$$\frac{r}{R} = \sum_{i=1}^{3} \theta_i$$

From: <Dan> To: <Sergei>

Dear Sergei, the r/R and sum of cosines is constant over all orbits!

From: <Sergei> To: <Dan>

Dear Dan, sorry to disappoint you: this identity holds in every triangle (without relation to billiards)

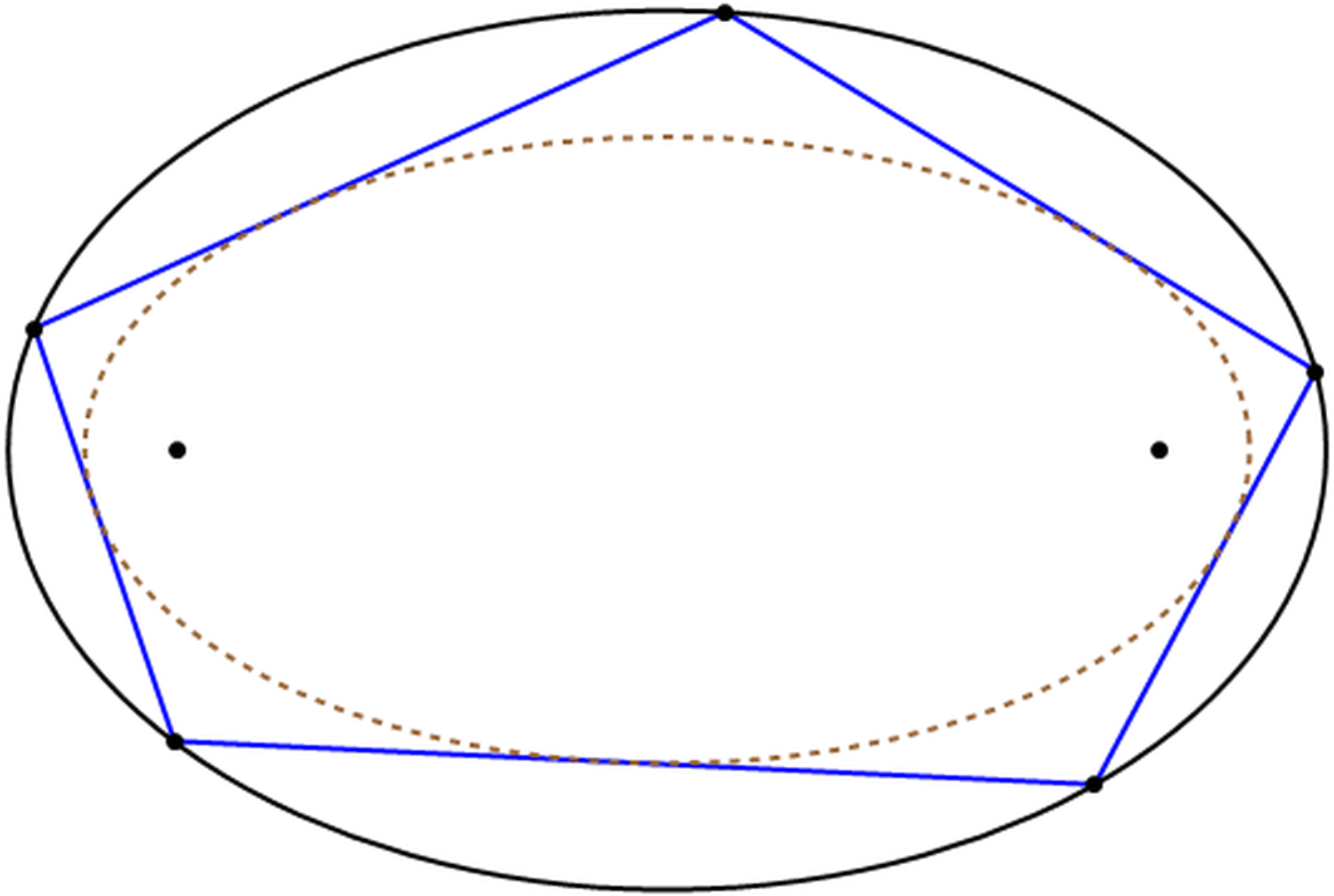
From: <Dan> To: <Sergei>

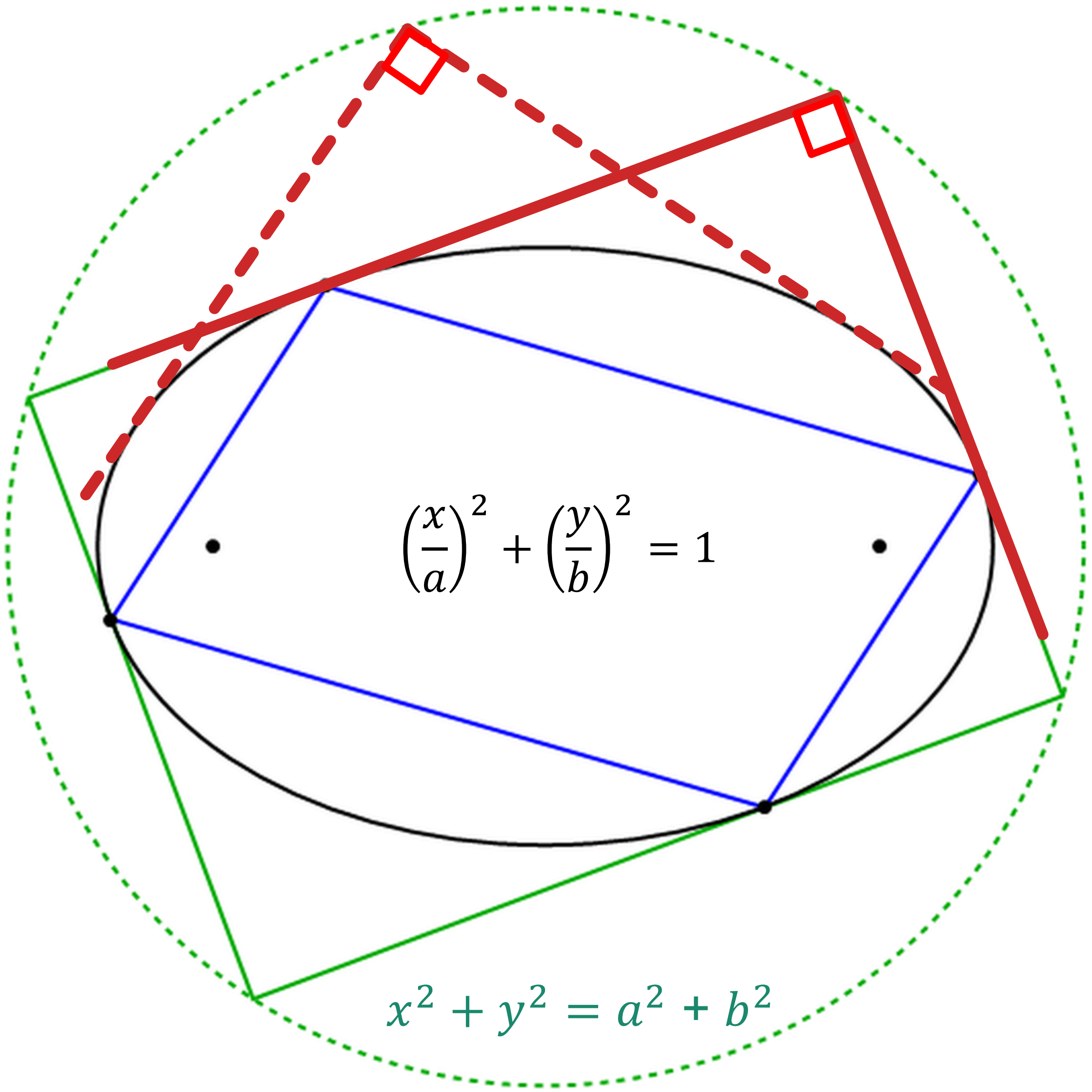
Dear Sergei, we were aware of the identity. But they are both staying constant over all orbits!

From: <Sergei> To: <Dan>

Oh, I see: the claim is that for a triangular orbit the sum of cosines is an integral. This is new to me.

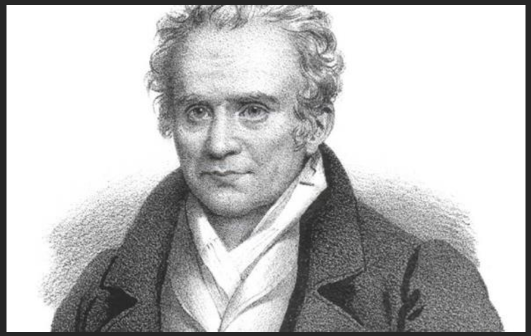
WHAT ABOUT N>3?





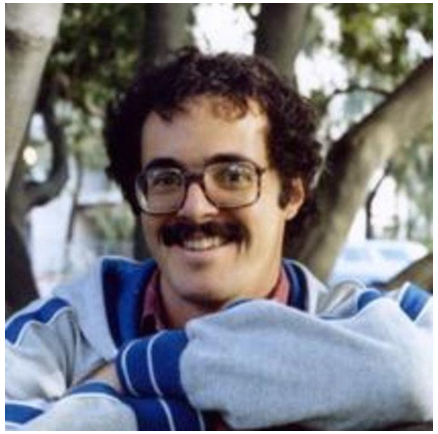
N=4: MONGE'S ORTHOPTIC CIRCLE

[VIDEO]



Gaspard Monge
(1746–1818)

CONVERSATION WITH JAIR MAY 29, 2019.



N=3: Triangle

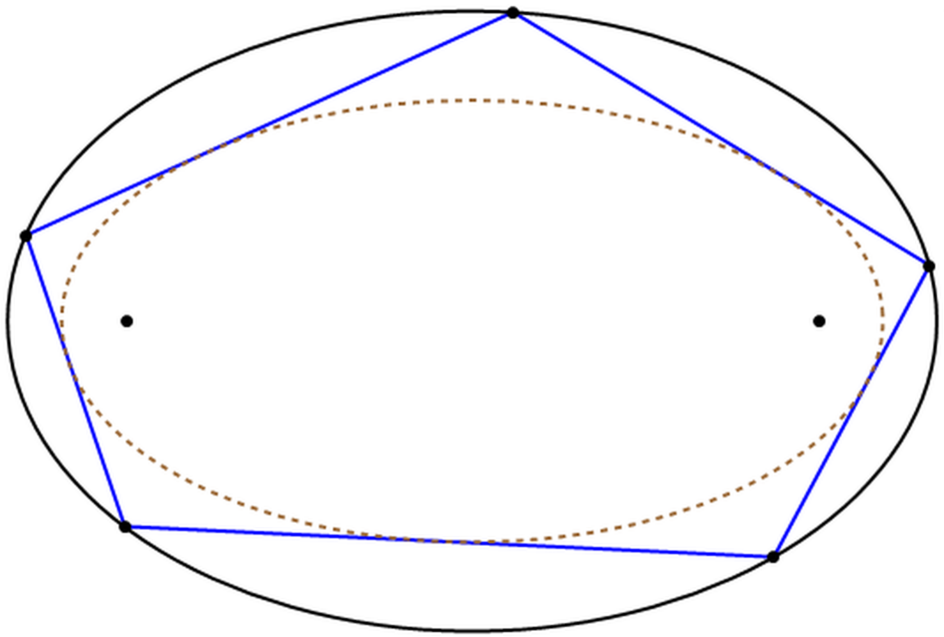
- r/R = constant
- r/R = 1+cosA+cosB+cosC

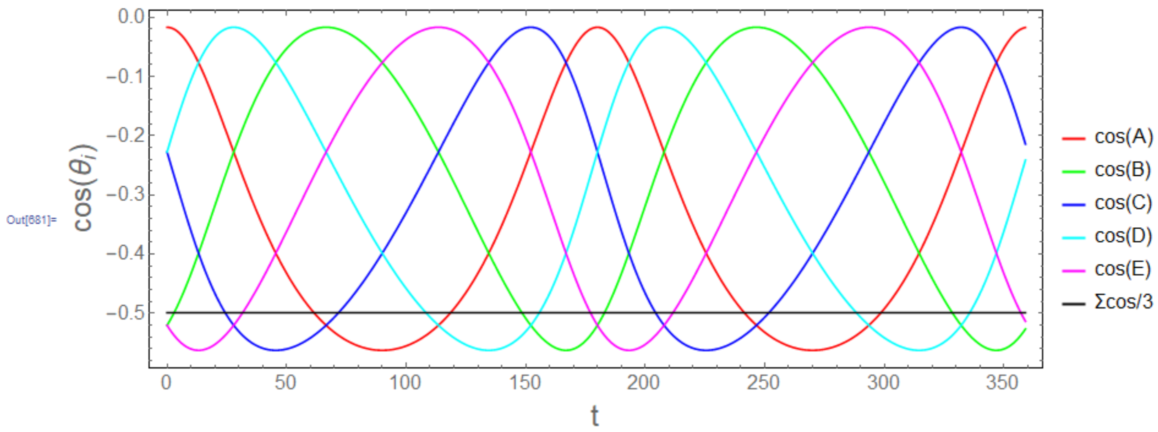
N=4: Parallelogram

- Consecutive angles supplementary:
  cosA=-cosB, cosC=-cosD
- So: cosA+cosB+cosC+cosD=0

N=5: Pentagon

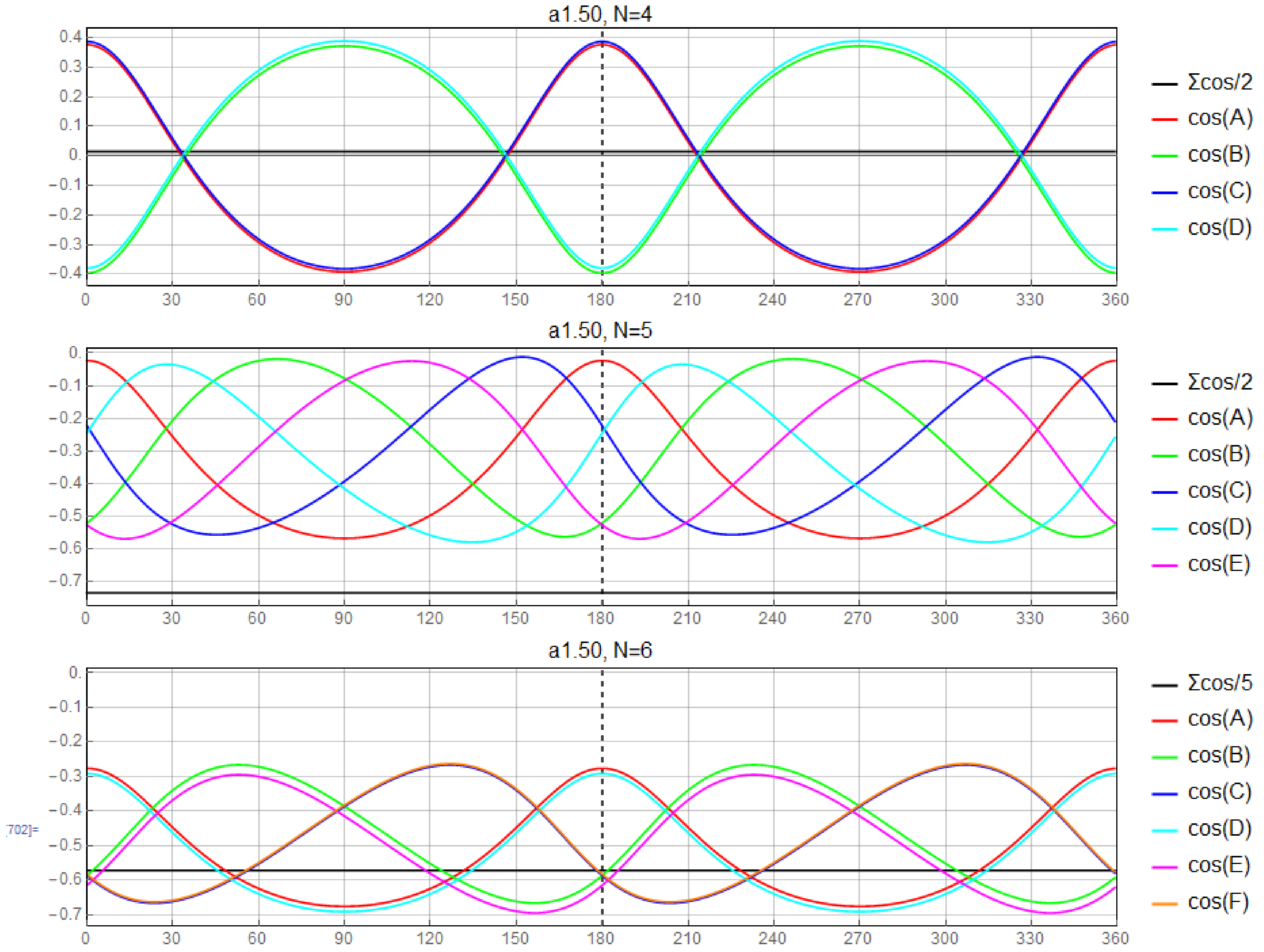
- Dan: Could it work?
- Jair: Try it!!!

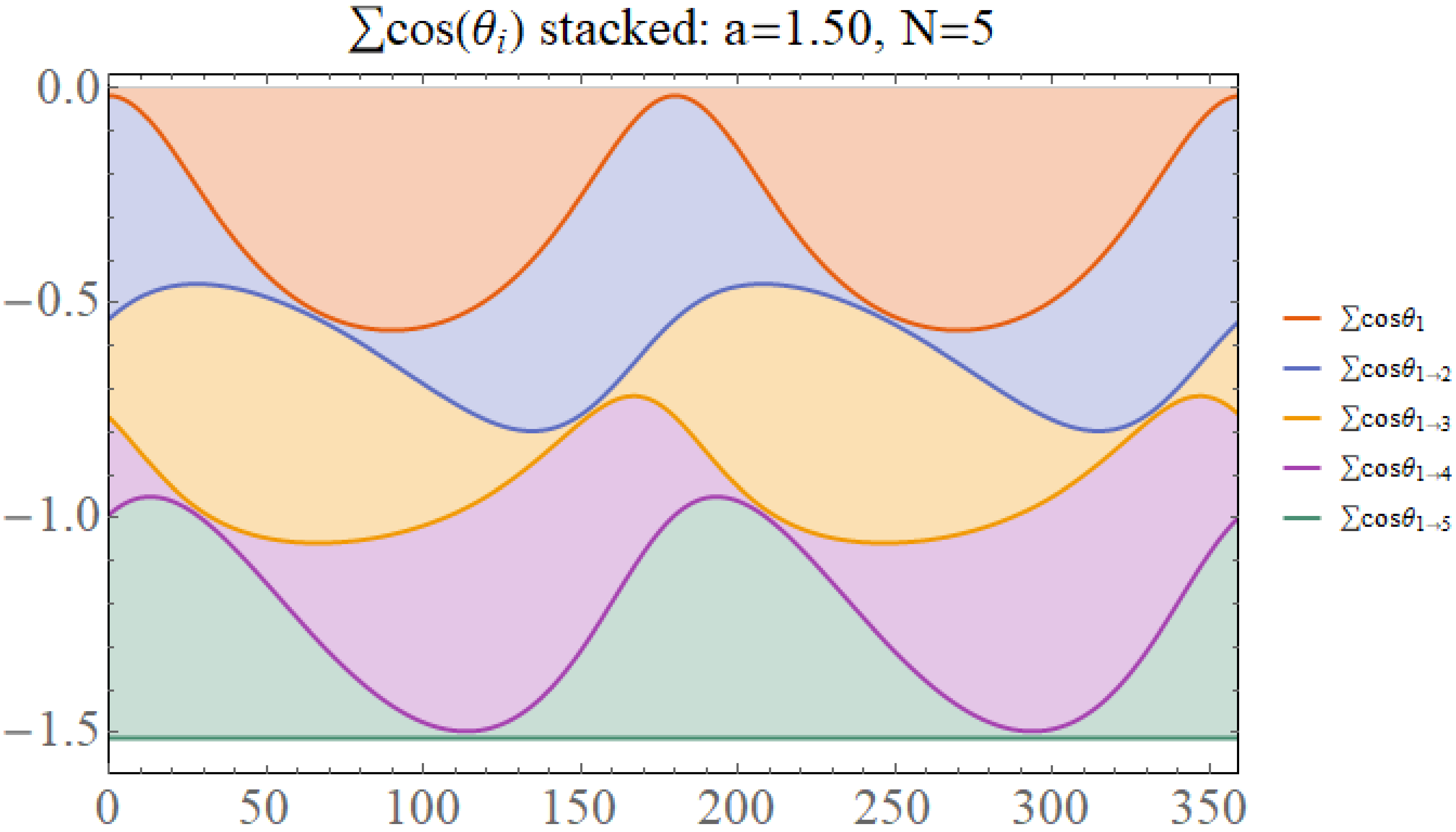



Out[681]=
cos(θ_i)
0.0
−0.1
−0.2
−0.3
−0.4
−0.5
0
50
100
150
200
250
300
350
t
cos(A)
cos(B)
cos(C)
cos(D)
cos(E)
Σcos/3

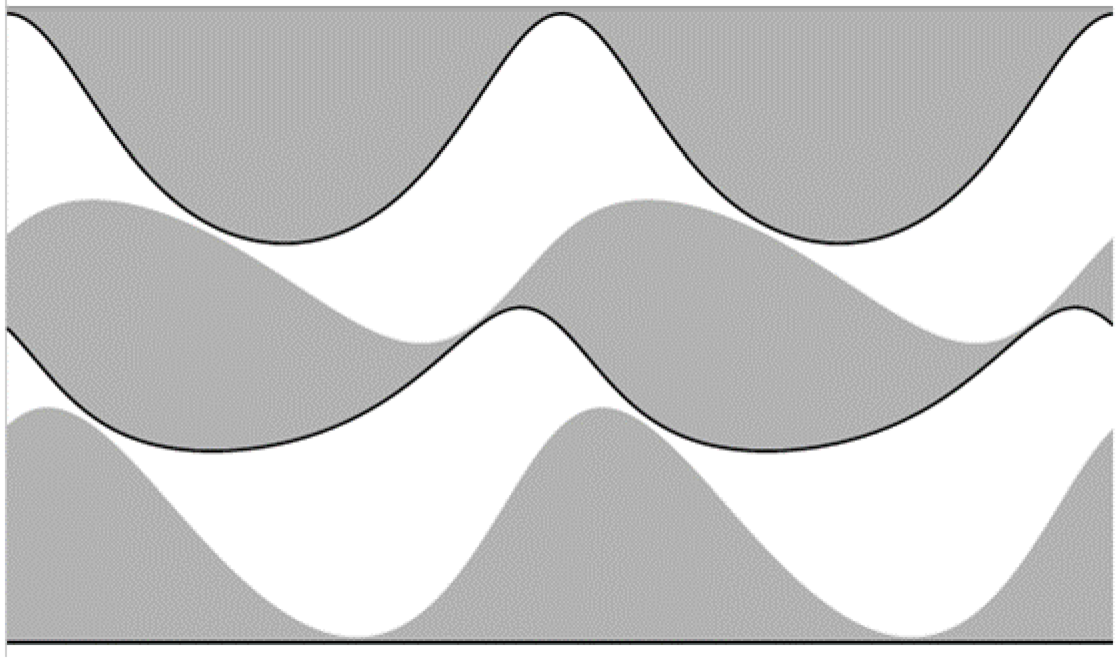
N=5 WORKS!!!

N=3~100 ALSO CONSERVES $\sum \cos\theta_i$





$\sum \cos(\theta_i)$ stacked: a=1.50, N=5
0.0
−0.5
−1.0
−1.5
0
50
100
150
200
250
300
350
$\sum \cos\theta_1$
$\sum \cos\theta_{1\to2}$
$\sum \cos\theta_{1\to3}$
$\sum \cos\theta_{1\to4}$
$\sum \cos\theta_{1\to5}$



JULY 3RD, 2019

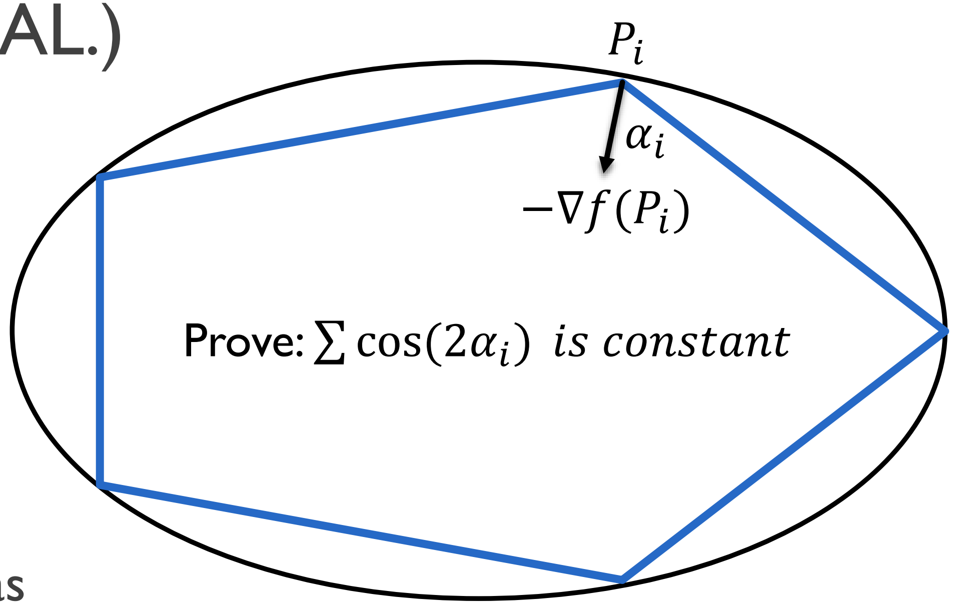
- From: <Sergei>, To: <Dan>

  Dear Dan: Richard and I **have a proof** that the sum of cosines is constant.
  Best, Sergei

- From: <Dan>, To: <Sergei>

  Wizardry!

SKETCH OF PROOF (TABACHNIKOV ET AL.)



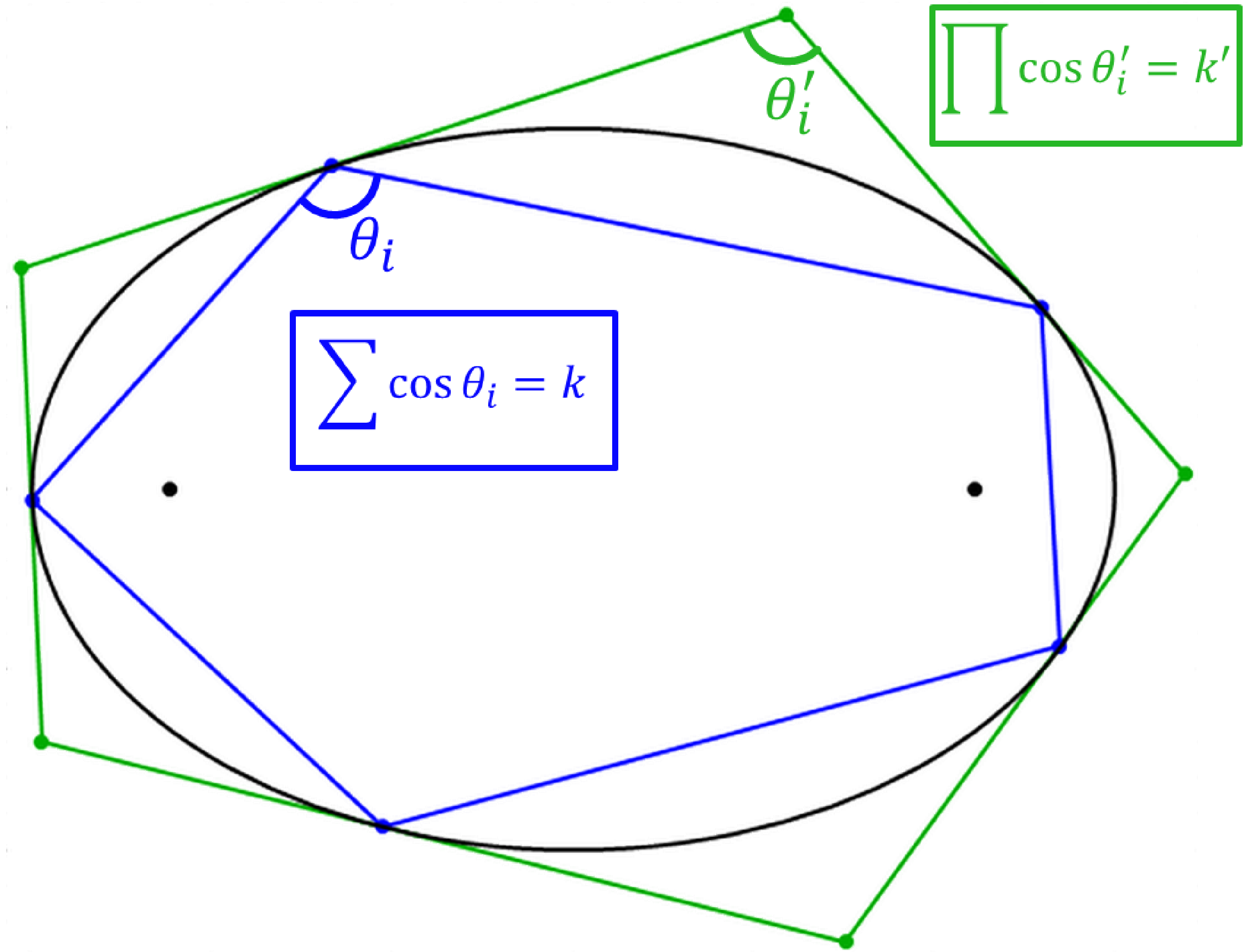
Joachimsthal

- $\cos\alpha = -\frac{\nabla f . \hat{v}}{|\nabla f|} = \frac{c}{|\nabla f|}$; $\cos(2\alpha) = 2\,(\cos\alpha)^2 - 1$
- $\sum \cos(2\alpha_i) = 2c \sum \frac{1}{|\nabla f(P_i)|^2} - N$
- Complexify and analyze the poles of $\sum \frac{1}{|\nabla f(P_i)|^2}$ over orbit as function on elliptic curve covering the conic
- Show complex poles of $\sum \frac{1}{|\nabla f(P_i)|^2}$ cancel each other

- So one has a meromorphic function without poles, hence a constant.
- In fact, there are $\frac{n-1}{2}$ conserved quantities.

JULY 25TH 2019

TANGENT POLYGON CONSERVES *PRODUCT* OF COSINES

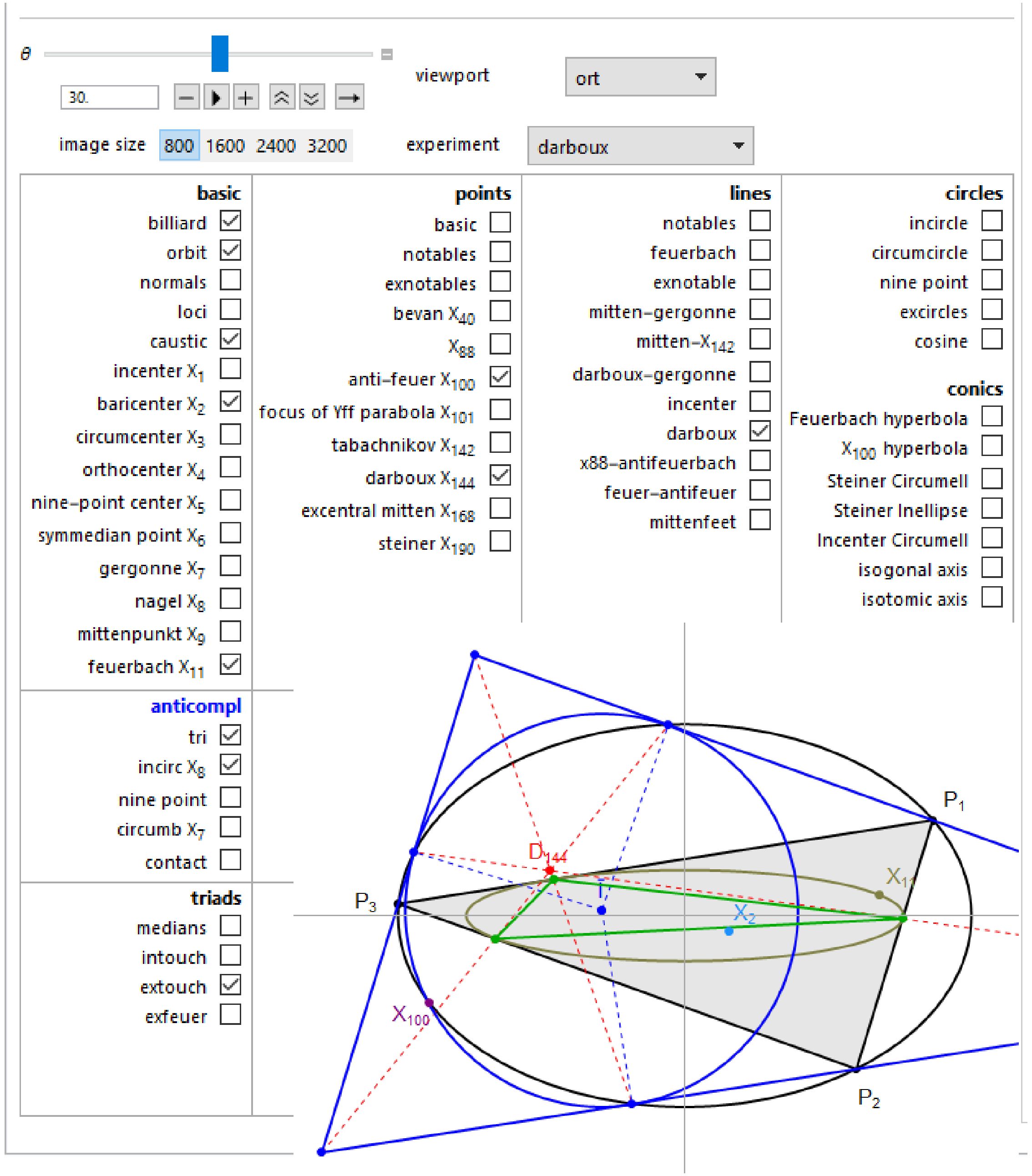


COMPUTATIONAL SETUP

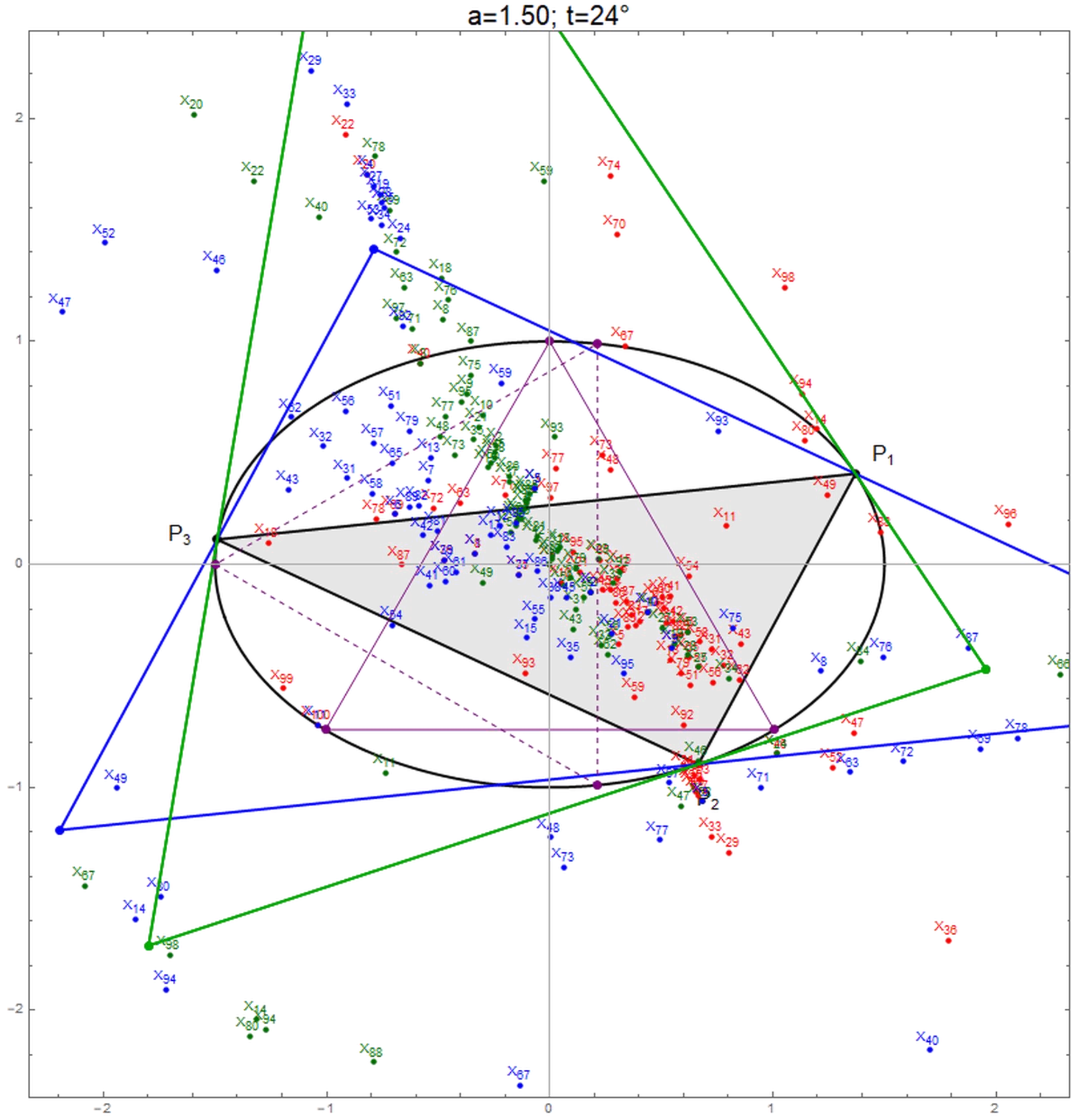
NUMERICAL PLAYGROUND

EXPERIMENTAL PLAYGROUND

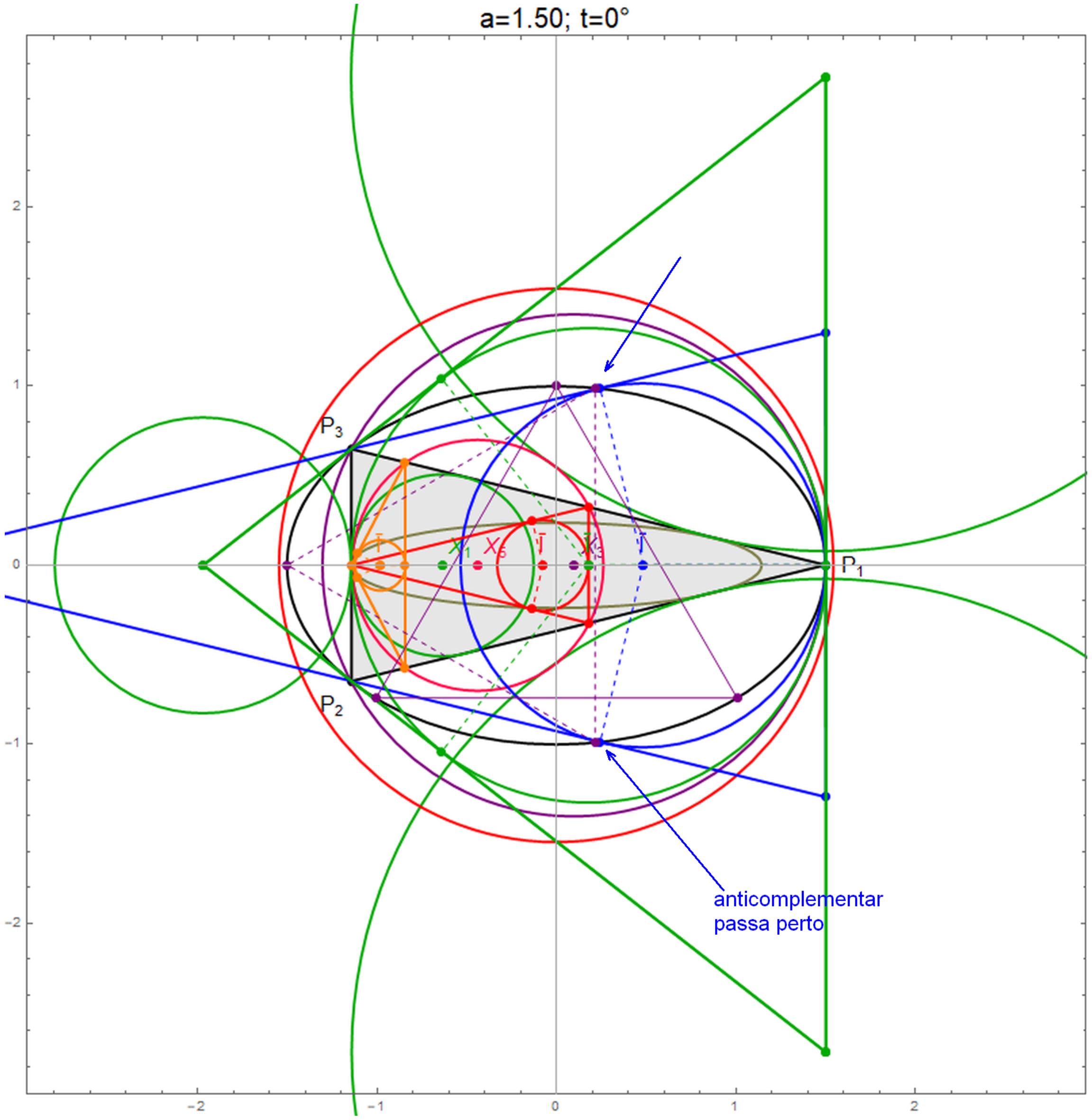
LINK

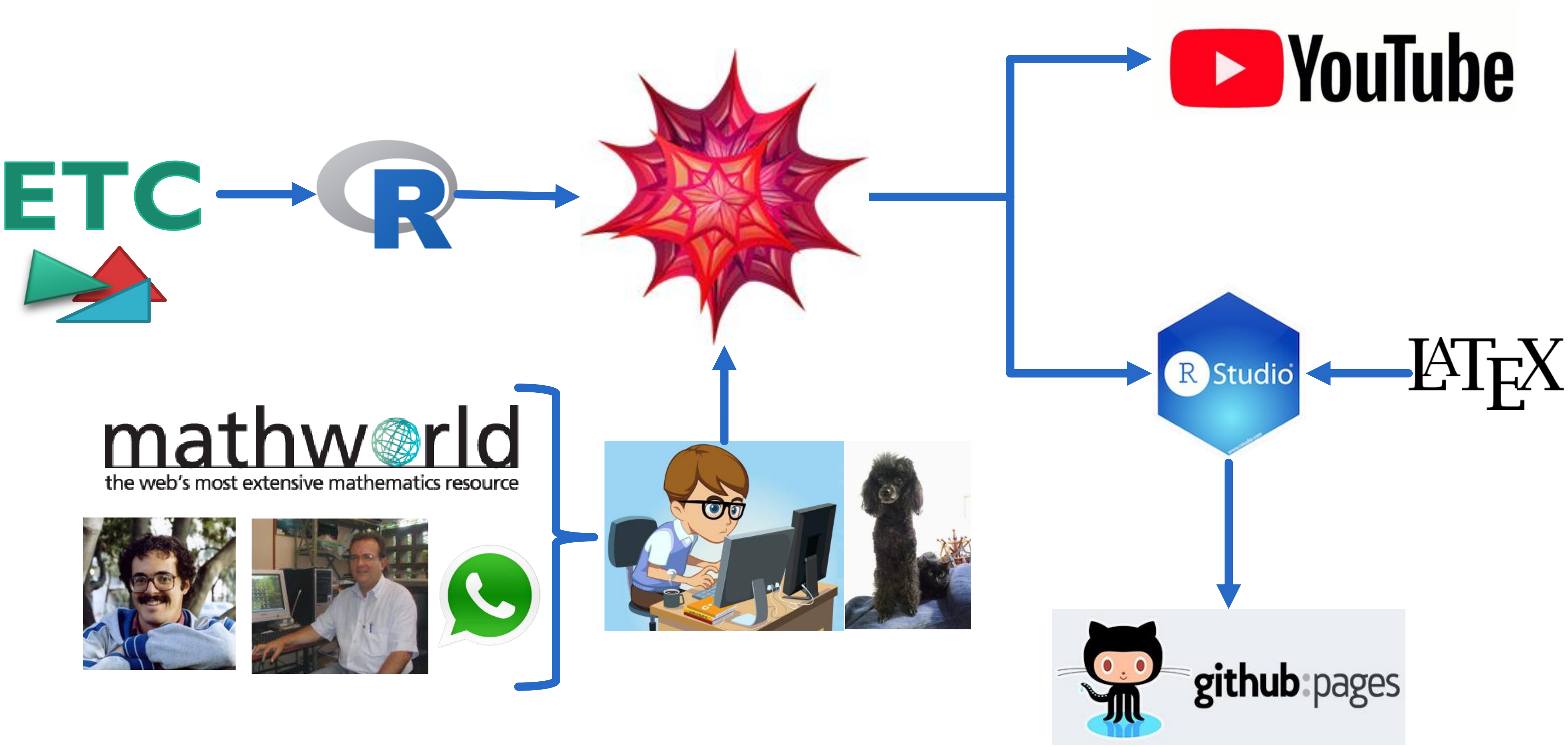


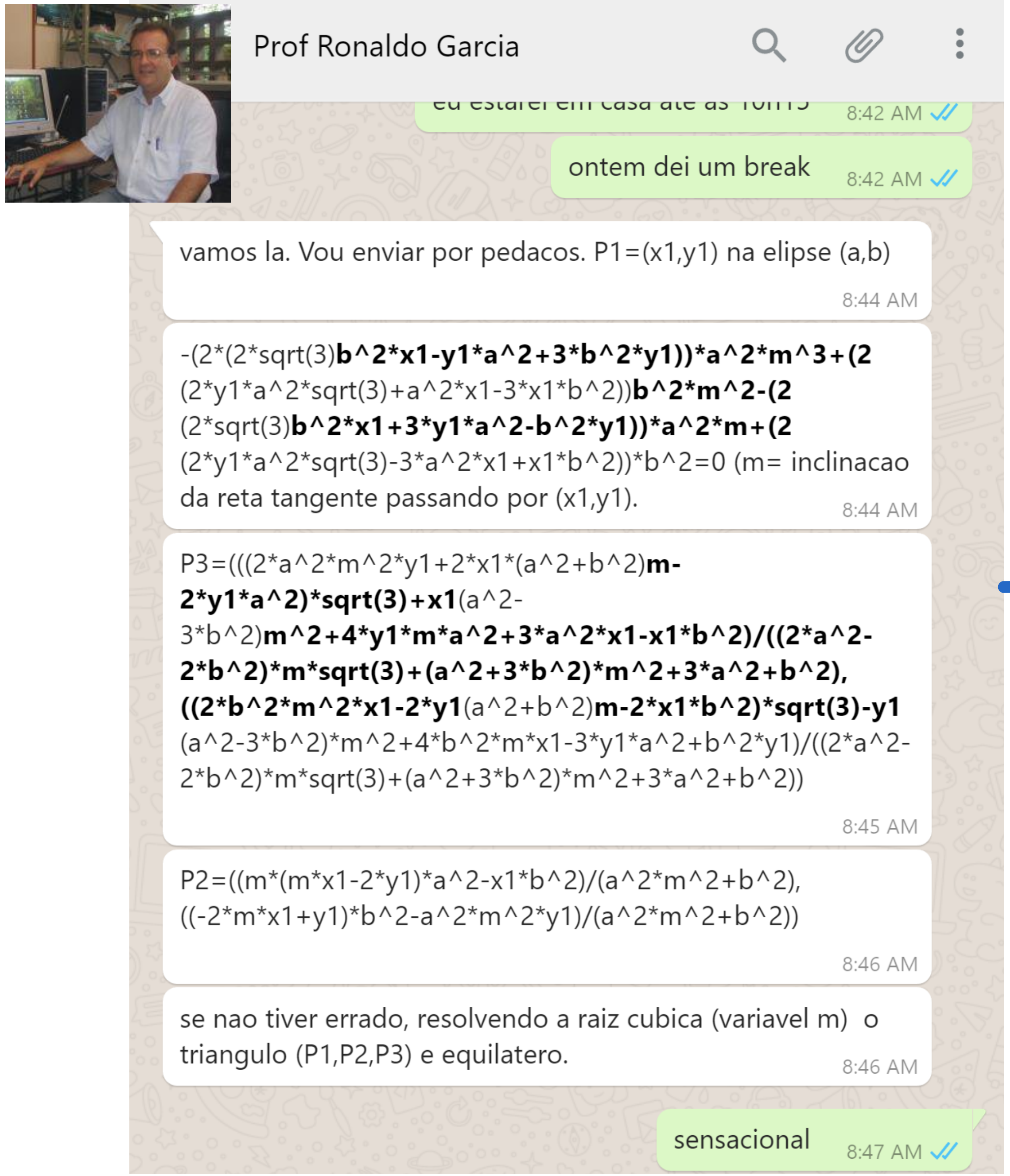
HUNTING I

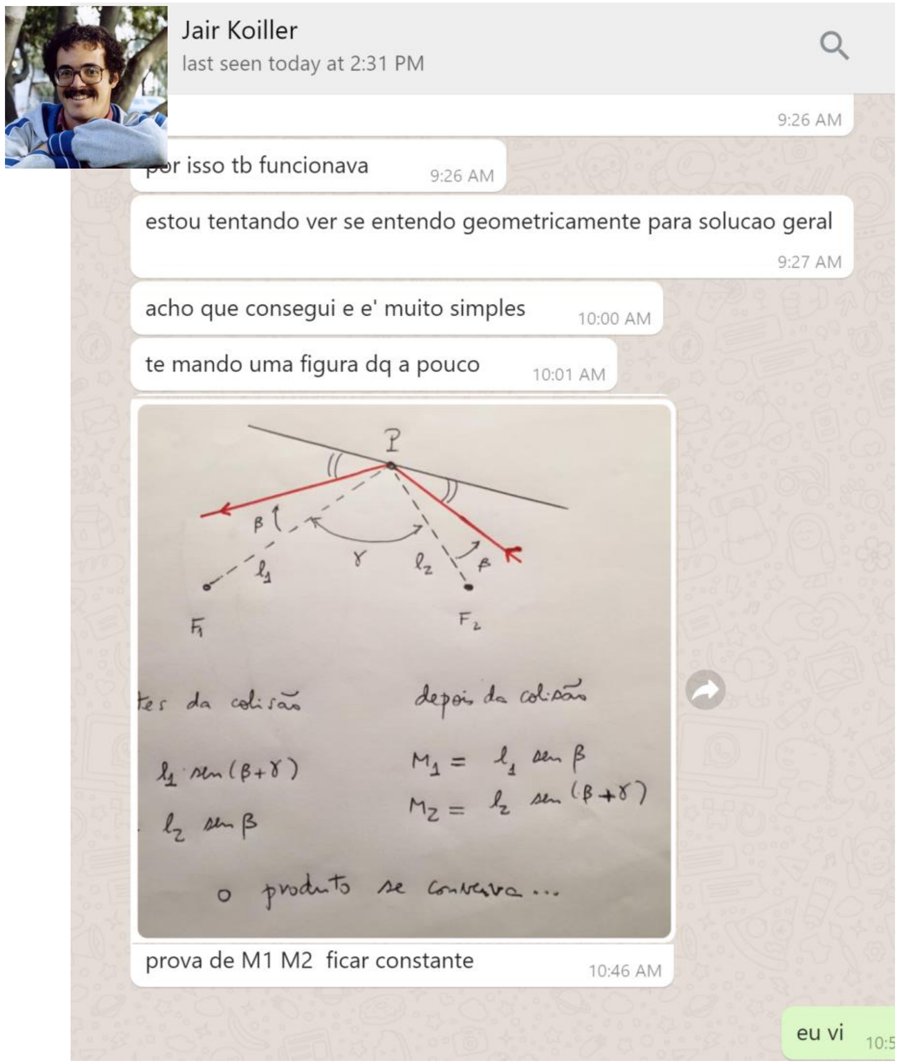


HUNTING 2




ETC
R
YouTube
R Studio
LaTeX
mathworld
the web's most extensive mathematics resource
github:pages


Prof Ronaldo Garcia
ontem dei um break 8:42 AM
vamos la. Vou enviar por pedacos. P1=(x1,y1) na elipse (a,b) 8:44 AM
-(2*(2*sqrt(3)b^2*x1-y1*a^2+3*b^2*y1))*a^2*m^3+(2(2*y1*a^2*sqrt(3)+a^2*x1-3*x1*b^2))b^2*m^2-(2(2*sqrt(3)b^2*x1+3*y1*a^2-b^2*y1))*a^2*m+(2(2*y1*a^2*sqrt(3)-3*a^2*x1+x1*b^2))*b^2=0 (m= inclinacao da reta tangente passando por (x1,y1). 8:44 AM
P3=(((2*a^2*m^2*y1+2*x1*(a^2+b^2)m-2*y1*a^2)*sqrt(3)+x1(a^2-3*b^2)m^2+4*y1*m*a^2+3*a^2*x1-x1*b^2)/((2*a^2-2*b^2)*m*sqrt(3)+(a^2+3*b^2)*m^2+3*a^2+b^2),((2*b^2*m^2*x1-2*y1(a^2+b^2)m-2*x1*b^2)*sqrt(3)-y1(a^2-3*b^2)*m^2+4*b^2*m*x1-3*y1*a^2+b^2*y1)/((2*a^2-2*b^2)*m*sqrt(3)+(a^2+3*b^2)*m^2+3*a^2+b^2)) 8:45 AM
P2=((m*(m*x1-2*y1)*a^2-x1*b^2)/(a^2*m^2+b^2),((-2*m*x1+y1)*b^2-a^2*m^2*y1)/(a^2*m^2+b^2)) 8:46 AM
se nao tiver errado, resolvendo a raiz cubica (variavel m) o triangulo (P1,P2,P3) e equilatero. 8:46 AM
sensacional 8:47 AM



Jair Koiller
last seen today at 2:31 PM
9:26 AM
por isso tb funcionava 9:26 AM
estou tentando ver se entendo geometricamente para solucao geral 9:27 AM
acho que consegui e e' muito simples 10:00 AM
te mando uma figura dq a pouco 10:01 AM
prova de M1 M2 ficar constante 10:46 AM
eu vi

MY LEARNING PROCESS

CURIOSITY

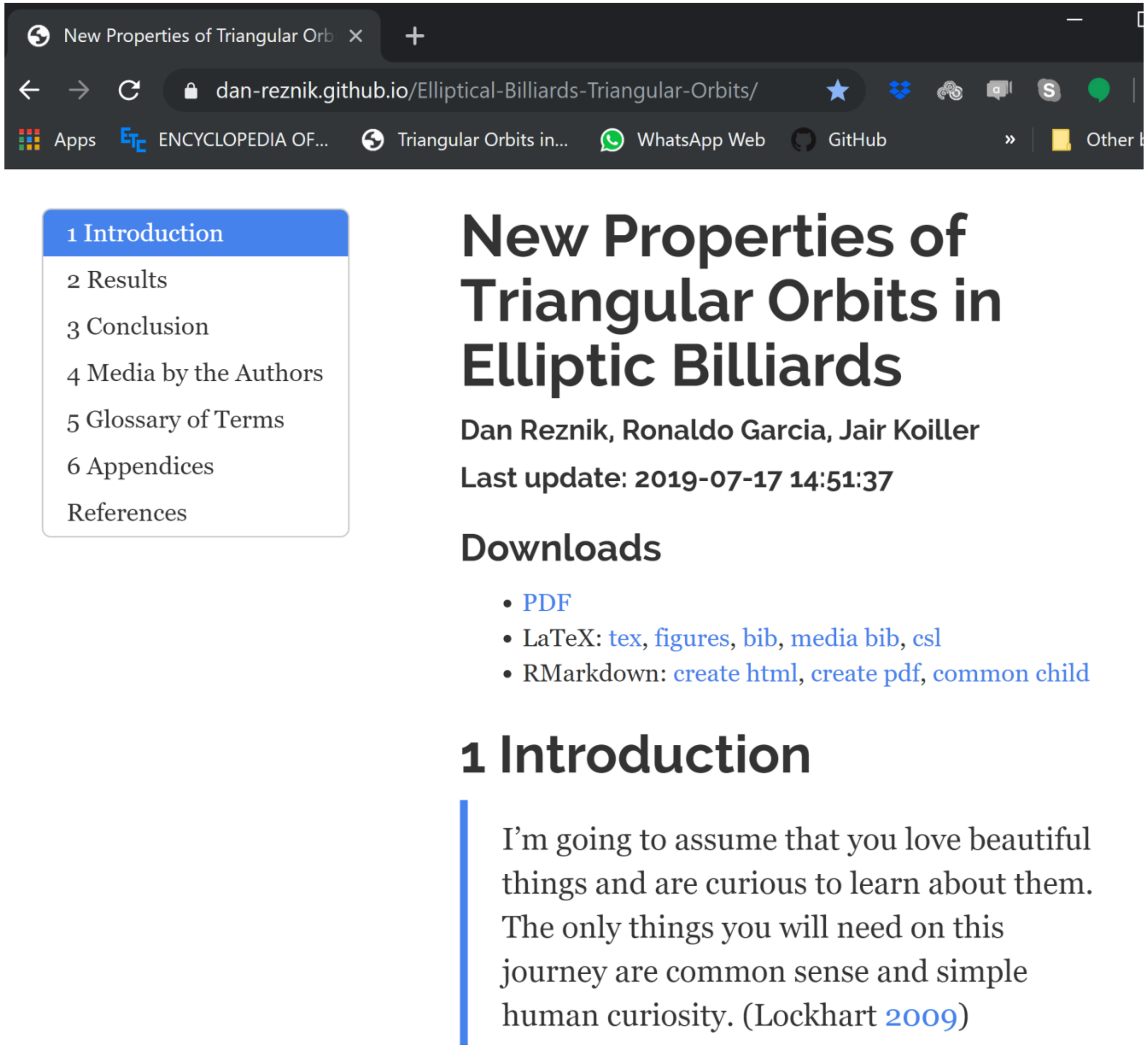
EXPERIMENTATION

LEARNING

(RE)DISCOVERING

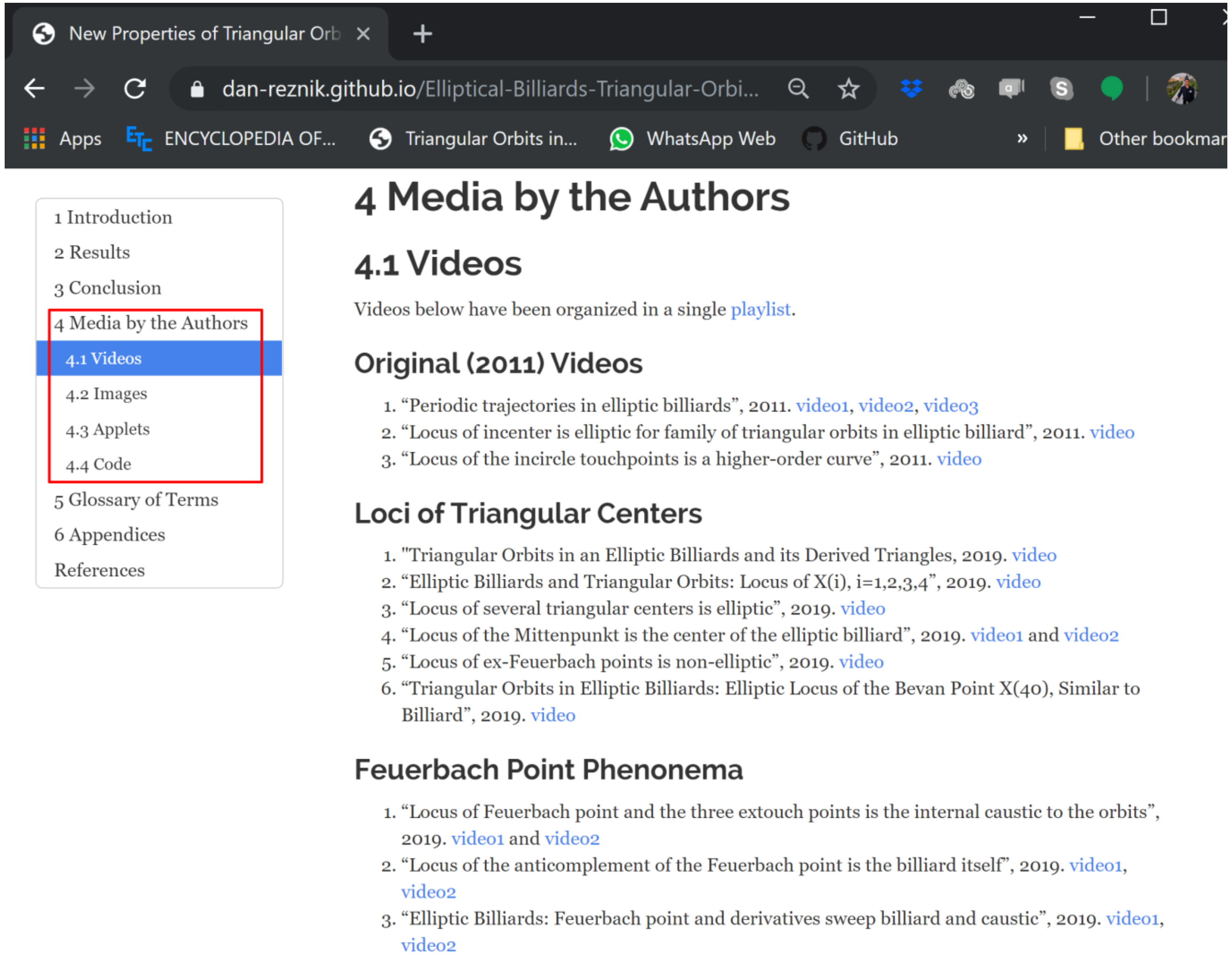
MAIN RESULTS & QUESTIONS

- Invariants for $\forall N$
  - "Mittenpunkt" stationary at billiard center
  - Integrals of motion:
    - $\sum \cos\theta_i = k$
    - $\prod \cos\theta_i' = k'$
- Open questions:
  - Elliptic vs non-elliptic loci?
  - Non-euclidean invariants?



OUR WEBPAGE:

BIT.LY/2JISAQ1


New Properties of Triangular Orb

dan-reznik.github.io/Elliptical-Billiards-Triangular-Orbi...

Apps ENCYCLOPEDIA OF... Triangular Orbits in... WhatsApp Web GitHub Other bookmar

4 Media by the Authors

4.1 Videos

Videos below have been organized in a single playlist.

Original (2011) Videos

1. "Periodic trajectories in elliptic billiards", 2011. video1, video2, video3
2. "Locus of incenter is elliptic for family of triangular orbits in elliptic billiard", 2011. video
3. "Locus of the incircle touchpoints is a higher-order curve", 2011. video

Loci of Triangular Centers

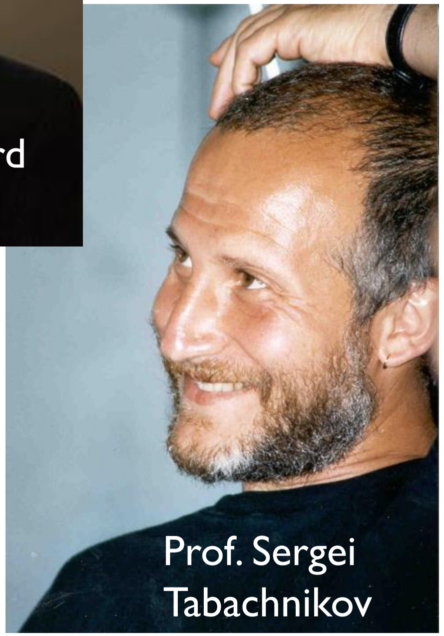
1. "Triangular Orbits in an Elliptic Billiards and its Derived Triangles, 2019. video
2. "Elliptic Billiards and Triangular Orbits: Locus of X(i), i=1,2,3,4", 2019. video
3. "Locus of several triangular centers is elliptic", 2019. video
4. "Locus of the Mittenpunkt is the center of the elliptic billiard", 2019. video1 and video2
5. "Locus of ex-Feuerbach points is non-elliptic", 2019. video
6. "Triangular Orbits in Elliptic Billiards: Elliptic Locus of the Bevan Point X(40), Similar to Billiard", 2019. video

Feuerbach Point Phenonema

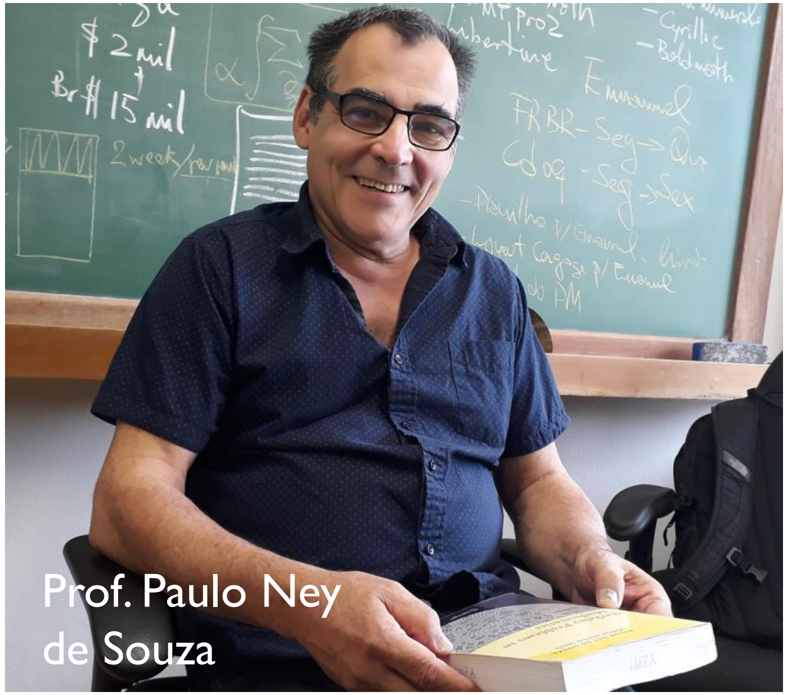
1. "Locus of Feuerbach point and the three extouch points is the internal caustic to the orbits", 2019. video1 and video2
2. "Locus of the anticomplement of the Feuerbach point is the billiard itself", 2019. video1, video2
3. "Elliptic Billiards: Feuerbach point and derivatives sweep billiard and caustic", 2019. video1, video2

MEDIA GALLERY LINK

THANKING MATHEMATICIANS



THANKING CO-AUTHORS / MENTORS


Prof. Paulo Ney de Souza

impa

Instituto de Matemática
Pura e Aplicada


Prof. Jorge P. Zubelli

THANKING FRIENDS AT IMPA

Jair: jairkoiller@gmail.com

Ronaldo: ragarcia@ufg.br

Dan: dreznik@gmail.com

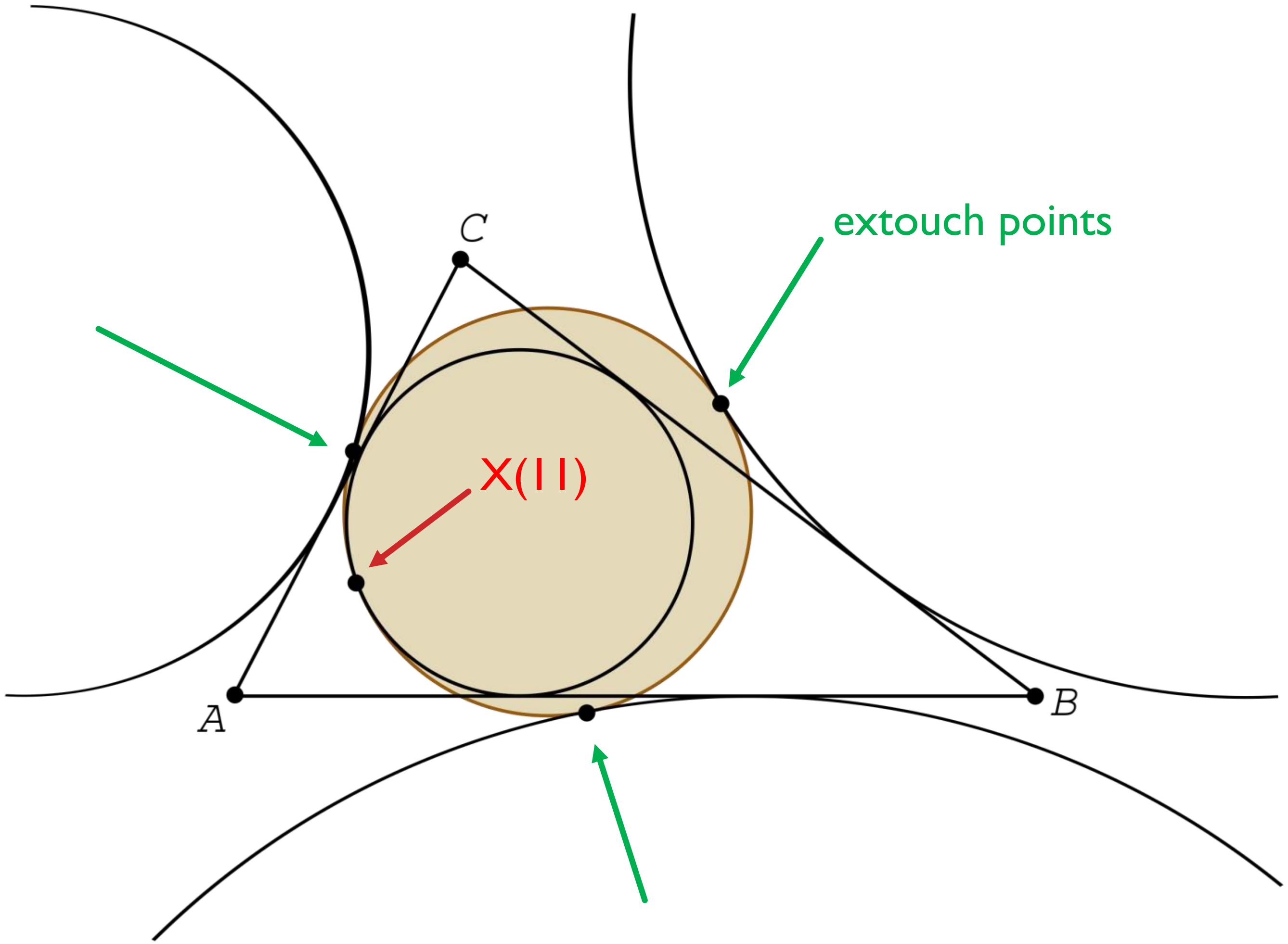
THANK YOU

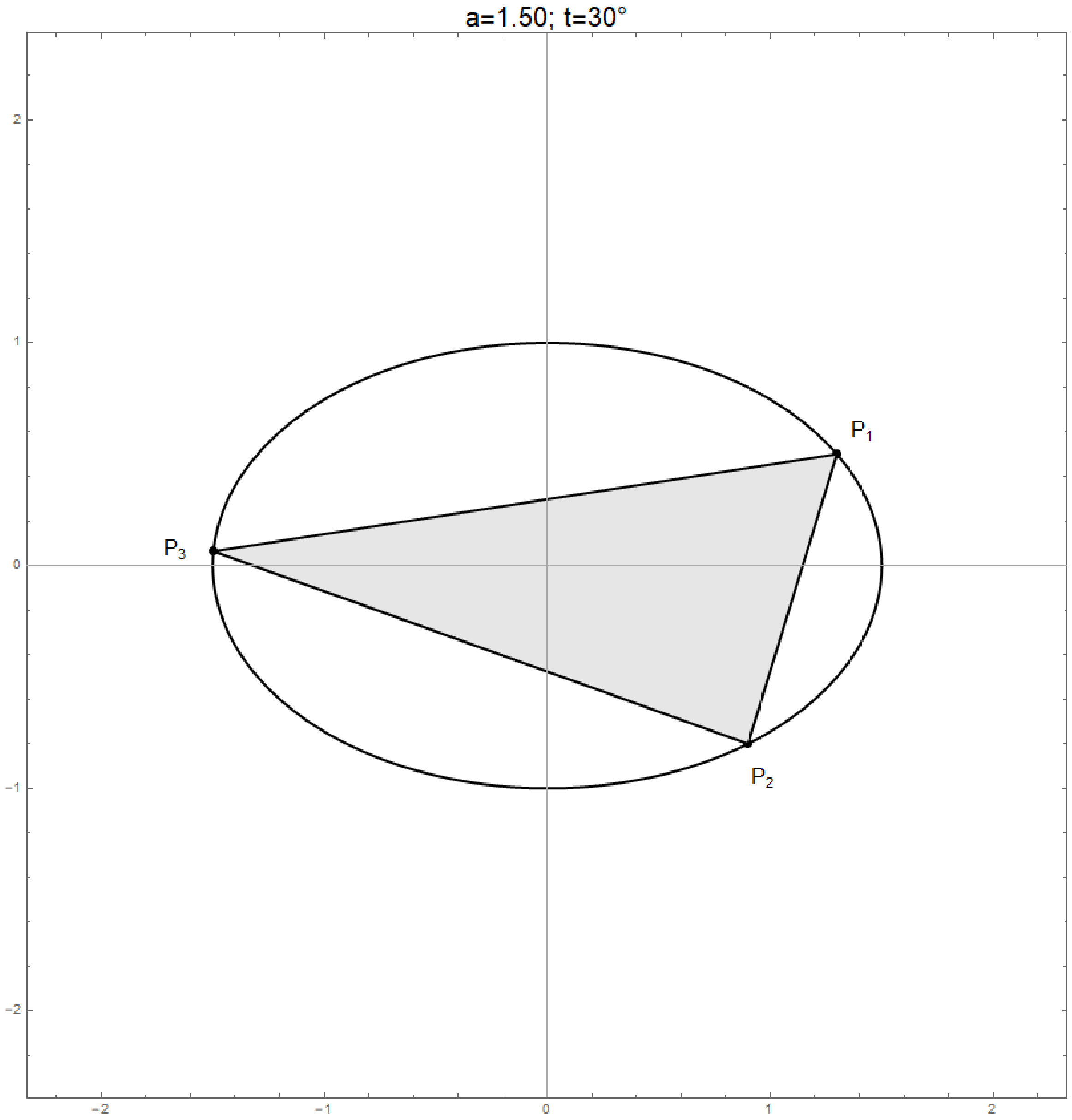
APPENDIX

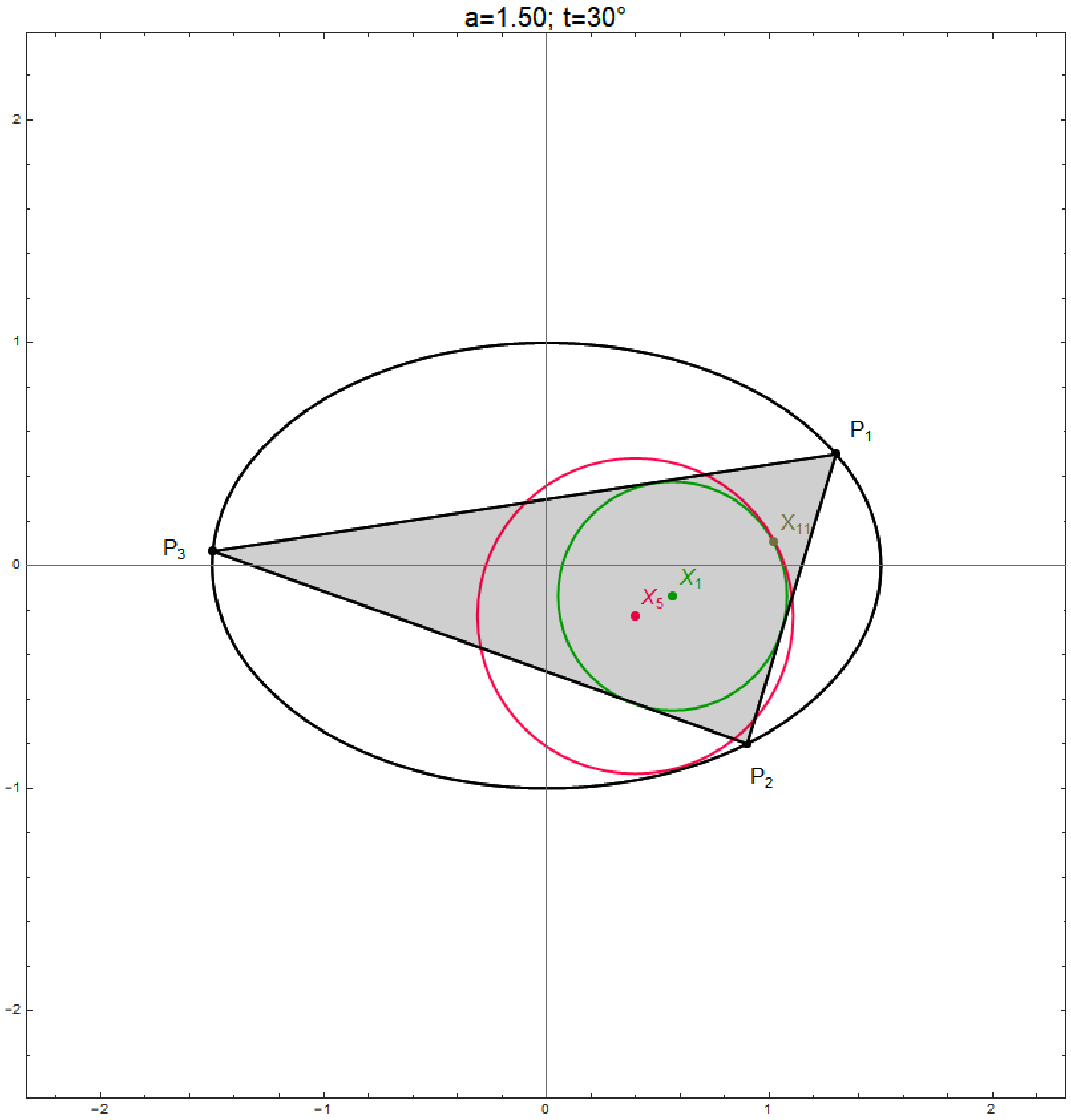
FEUERBACH PHENOMENA

Karl Wilhelm Feuerbach
1800-1834

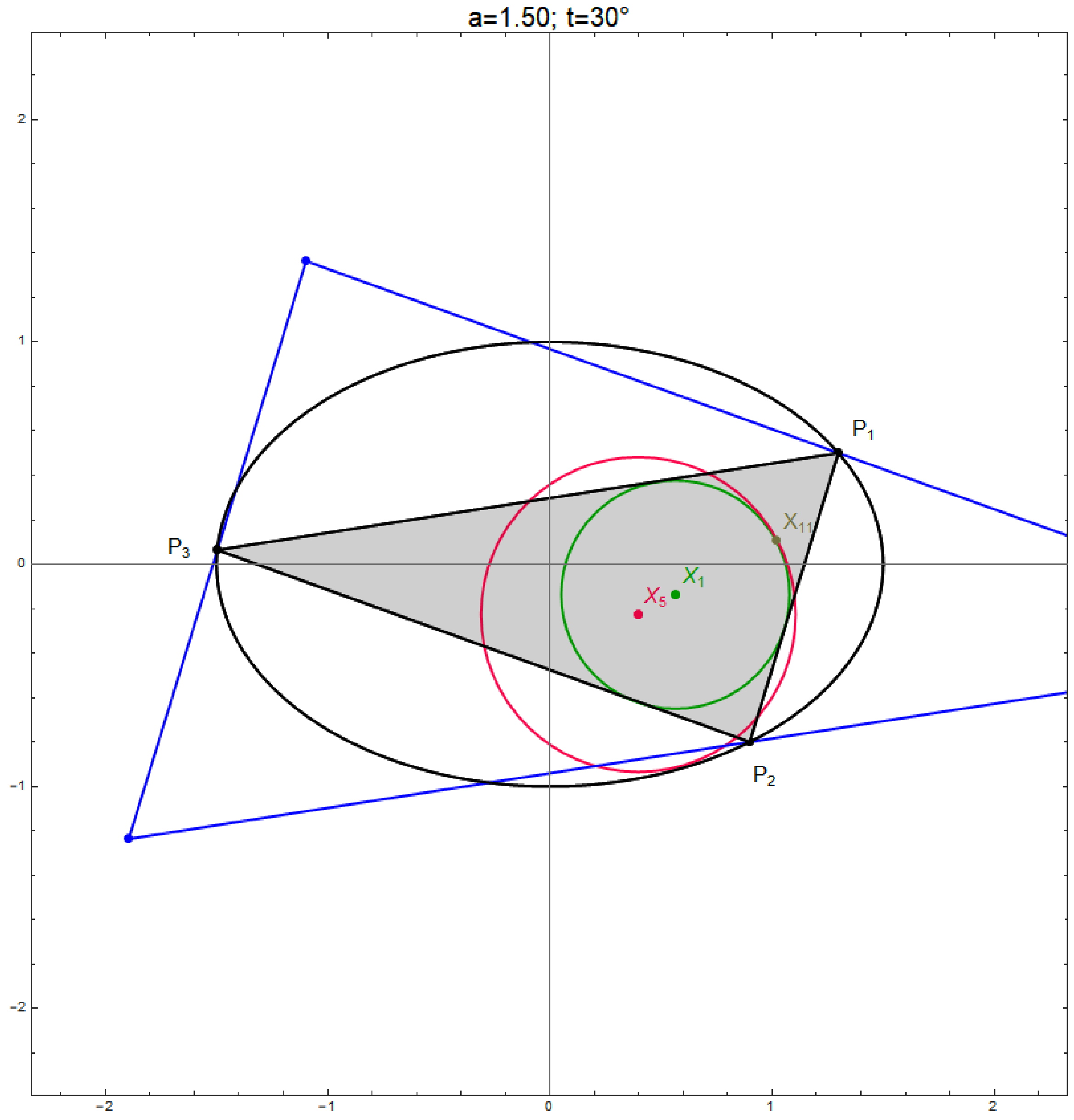
FEUERBACH'S THEOREM



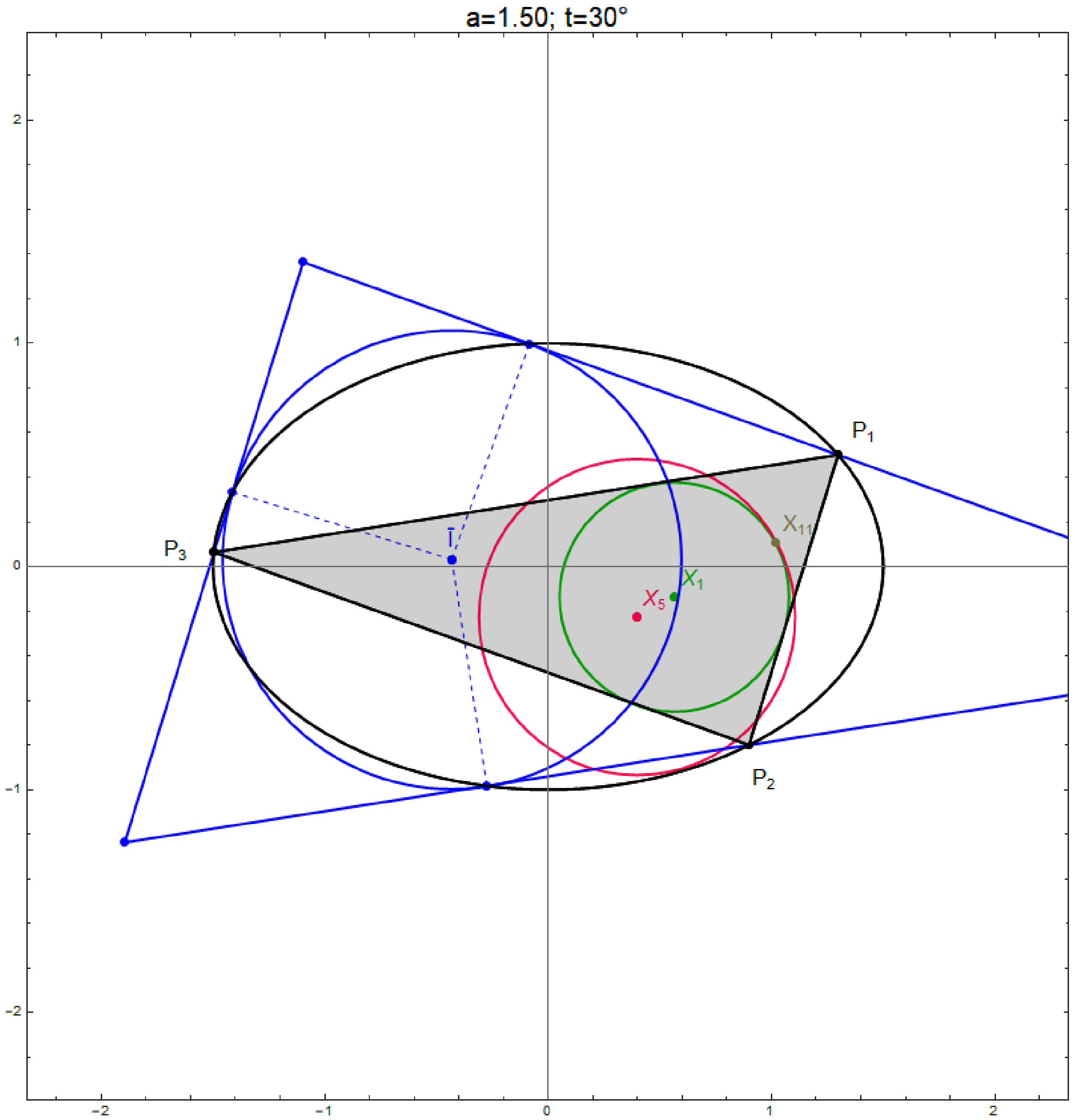

a=1.50; t=30°
P1
P2
P3
2
1
0
-1
-2
-2
-1
0
1
2



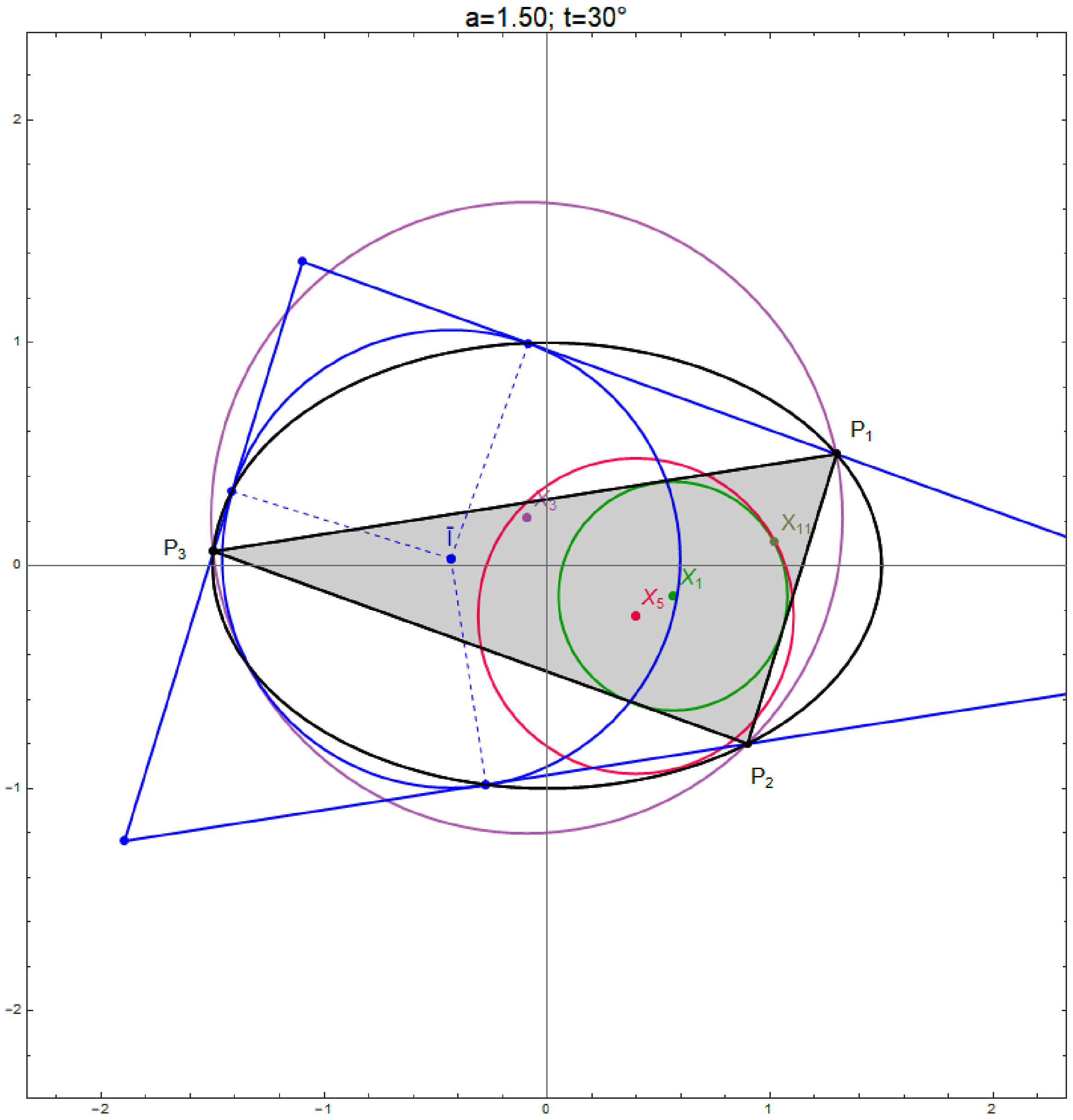
a=1.50; t=30°
P1
P2
P3
X11
X1
X5



a=1.50; t=30°
P1
P2
P3
X11
X1
X5



a=1.50; t=30°
P1
P2
P3
X11
X1
X5
Ī


a=1.50; t=30°
P1
P2
P3
X3
X11
X1
X5
I
2
1
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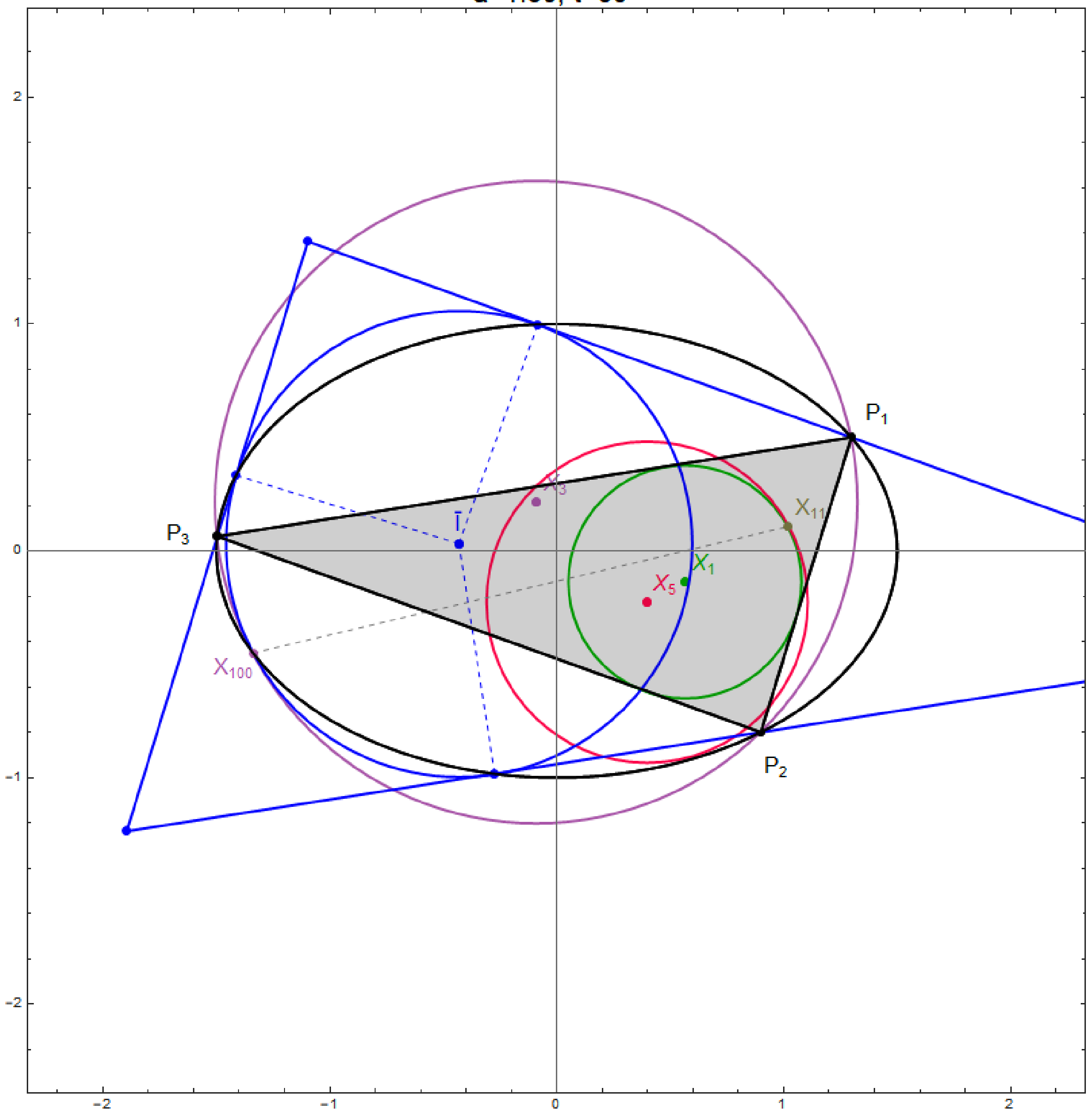
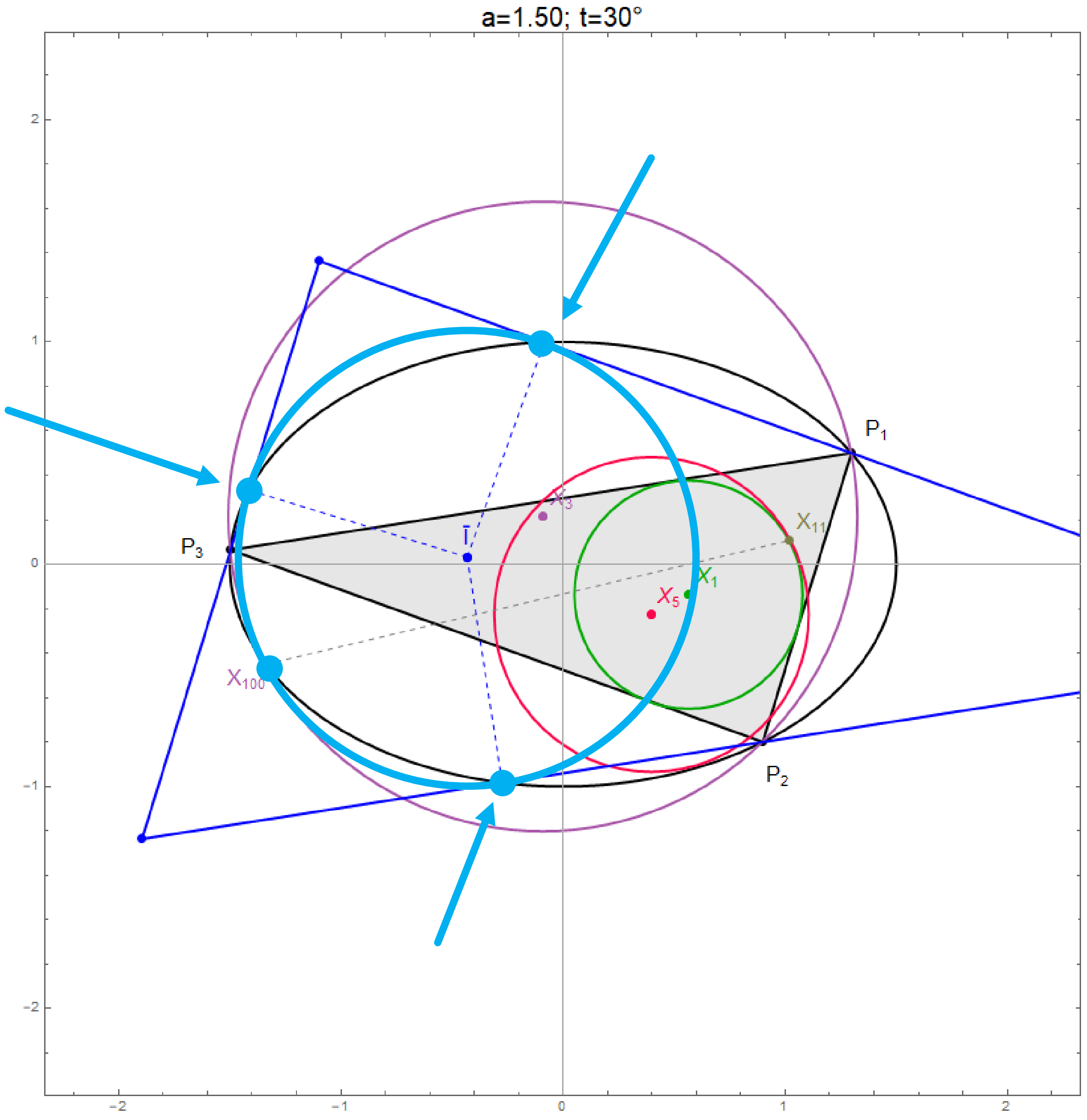
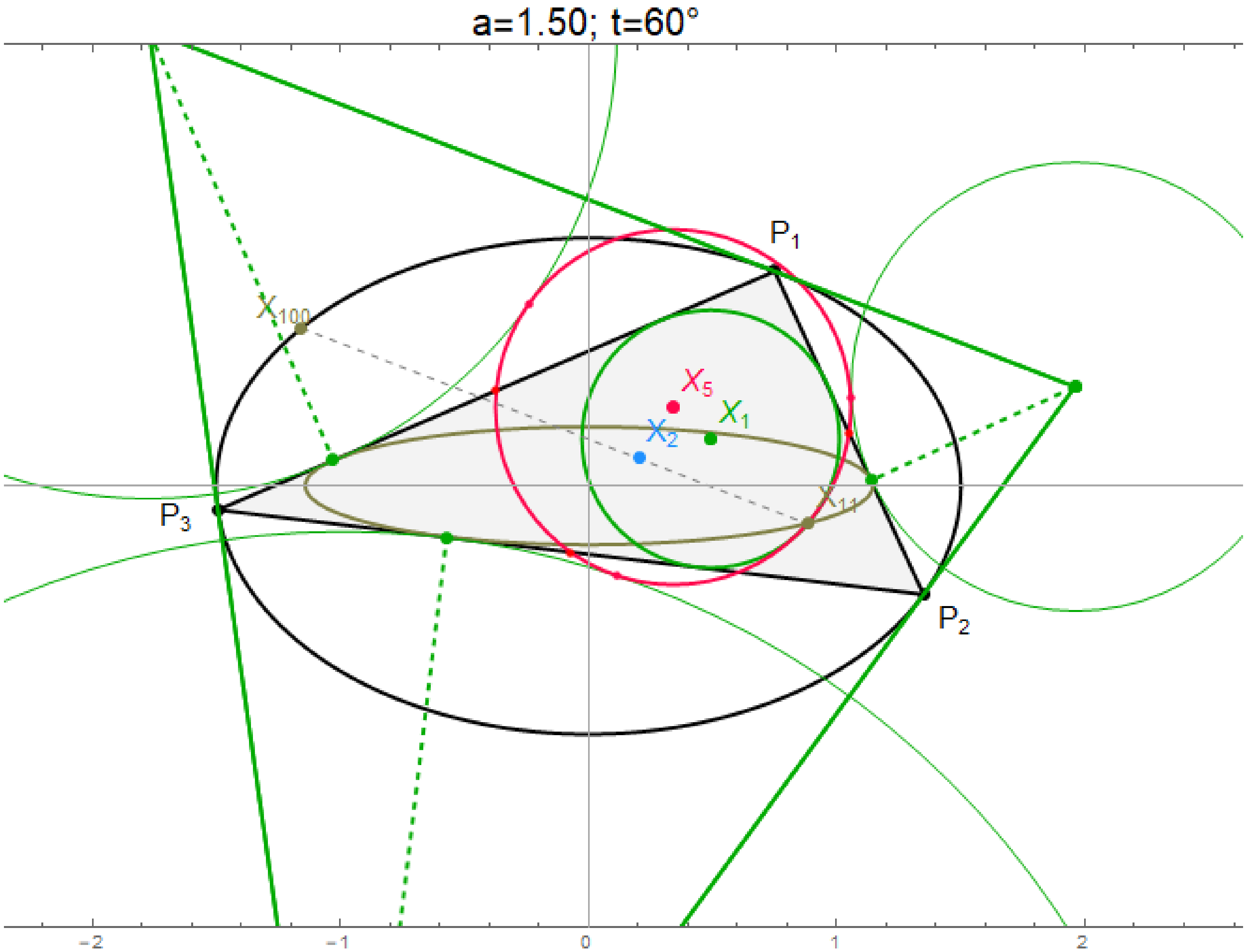
-2
-1
0
1
2

a=1.50; t=30°





Prof. Igor Minevich
Boston College



1) FEUERBACH POINT SWEEPS BILLIARD

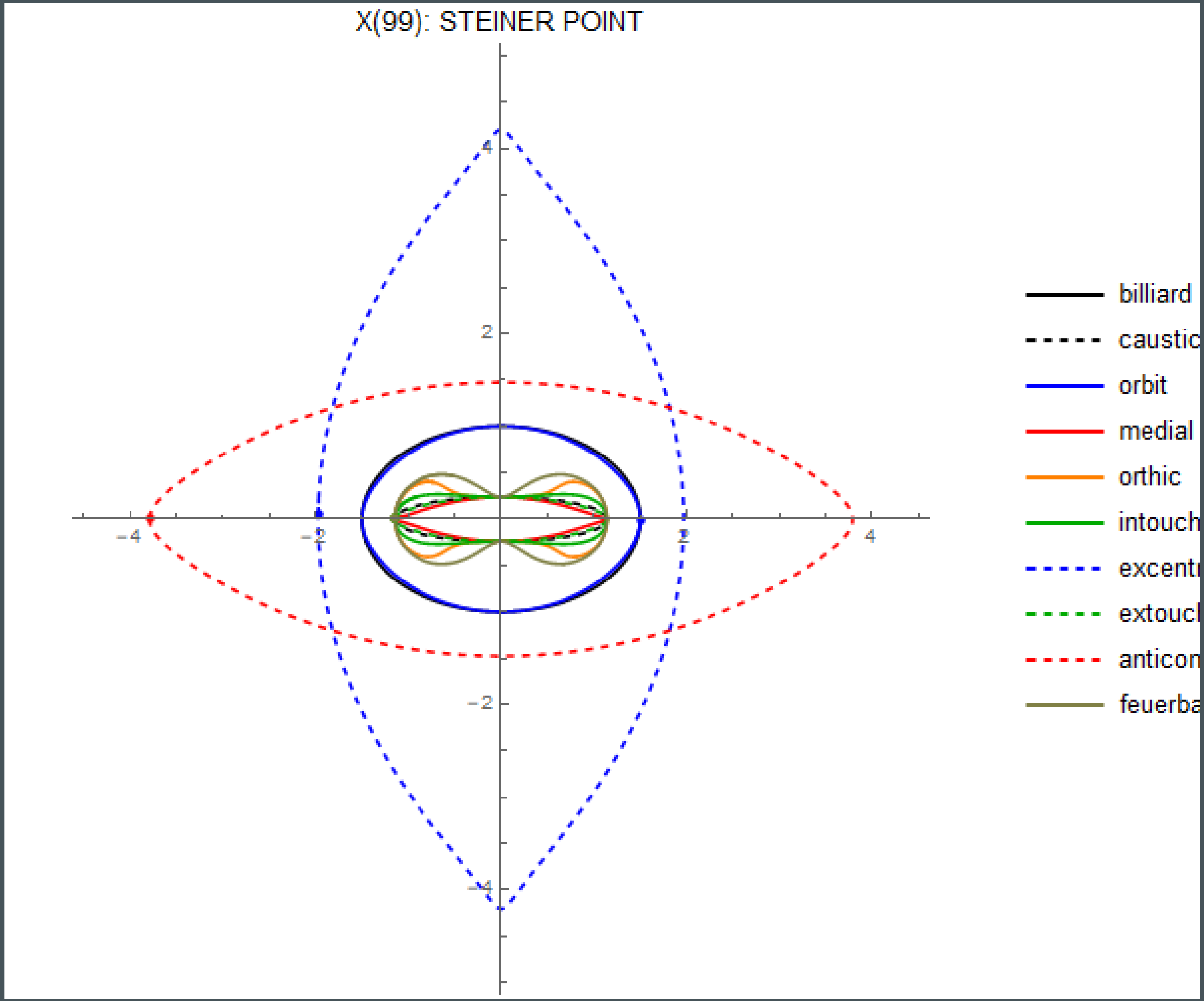
2) ANTICOMPLEMENT OF FEUERBACH SWEEPS BILLIARD

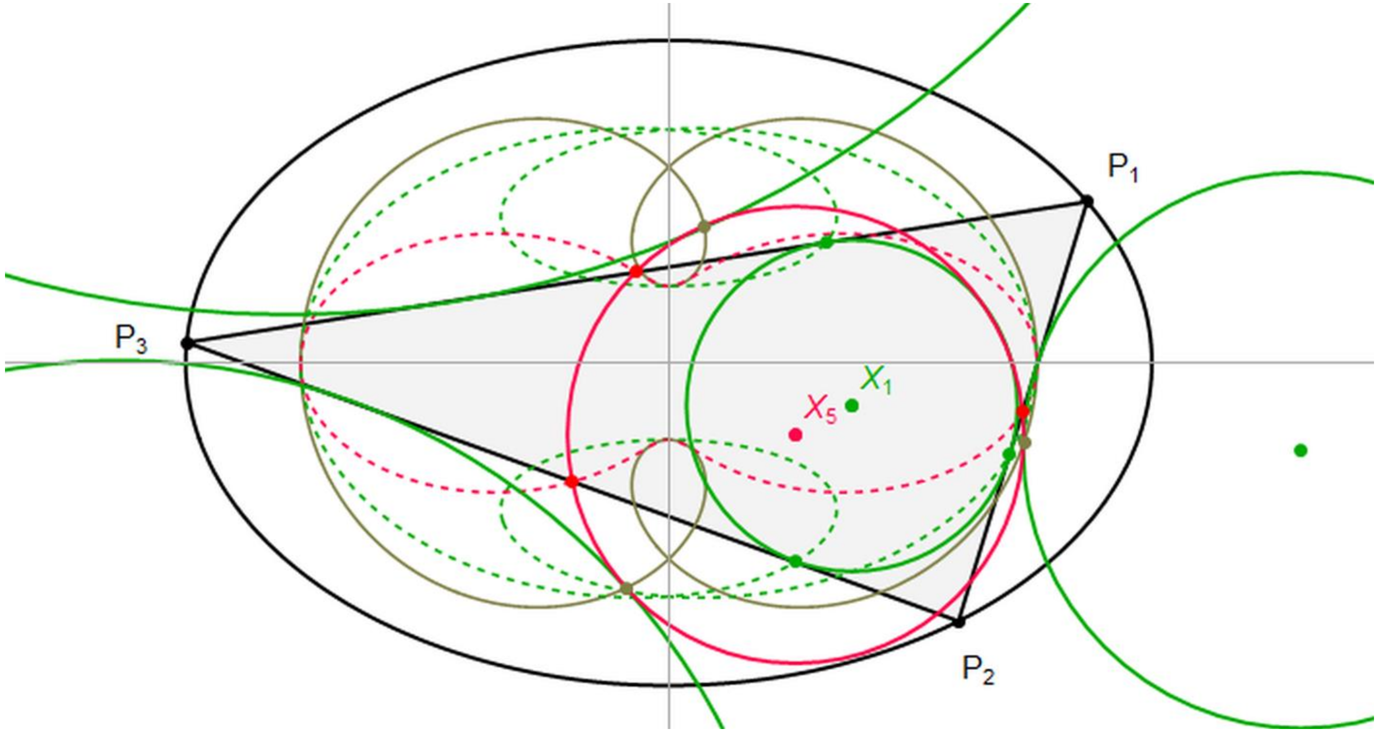
3) EXTOUCHPOINTS SWEEP BILLIARD (IN REVERSE!)

[VIDEO]

IN GENERAL LOCI ARE NON-ELLIPTIC

GALLERY W/ 700 LOCI



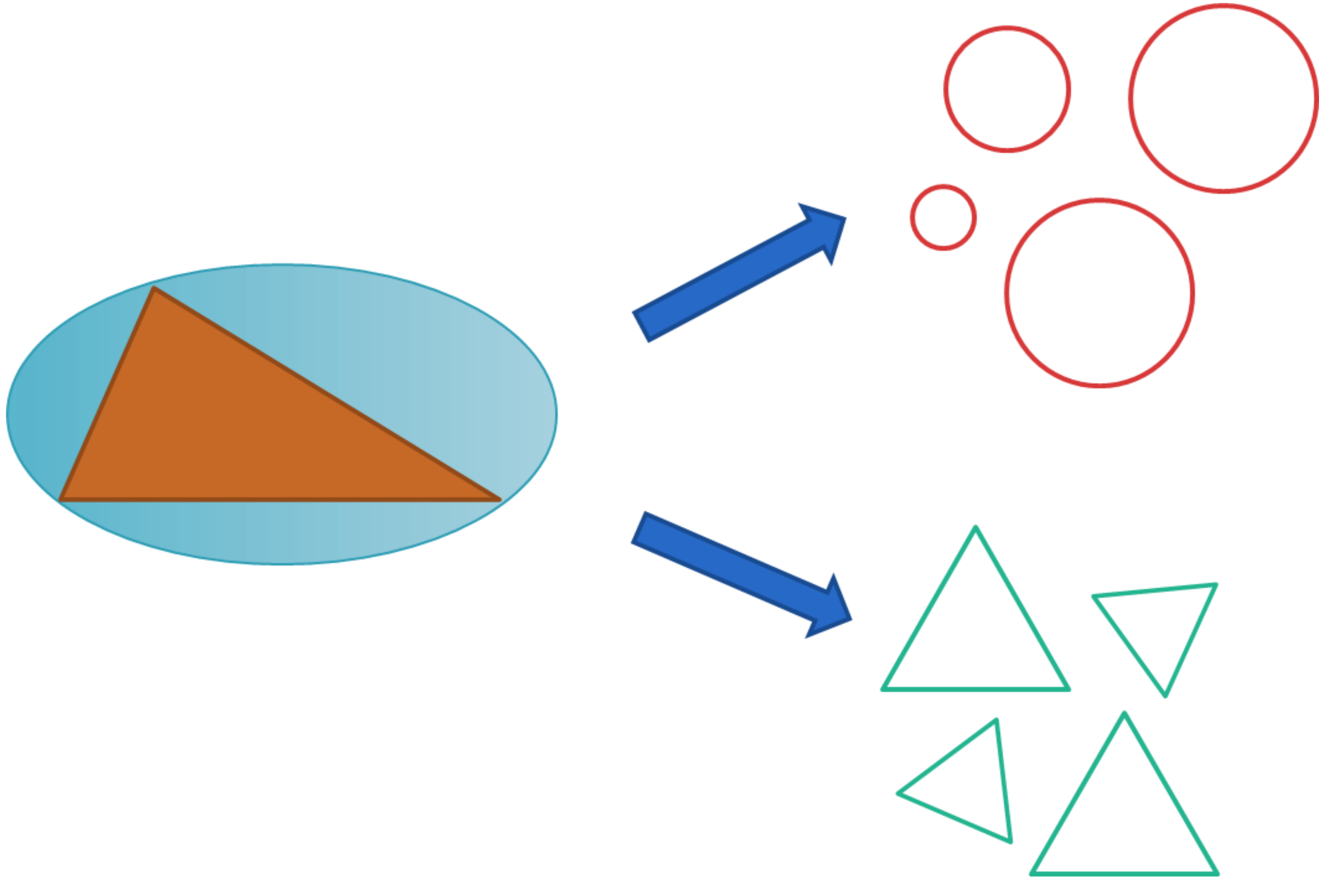


NON-ELLIPTIC LOCI:
1) MEDIAN
2) INTOUCH
3) FEUERBACH TRIANGLE

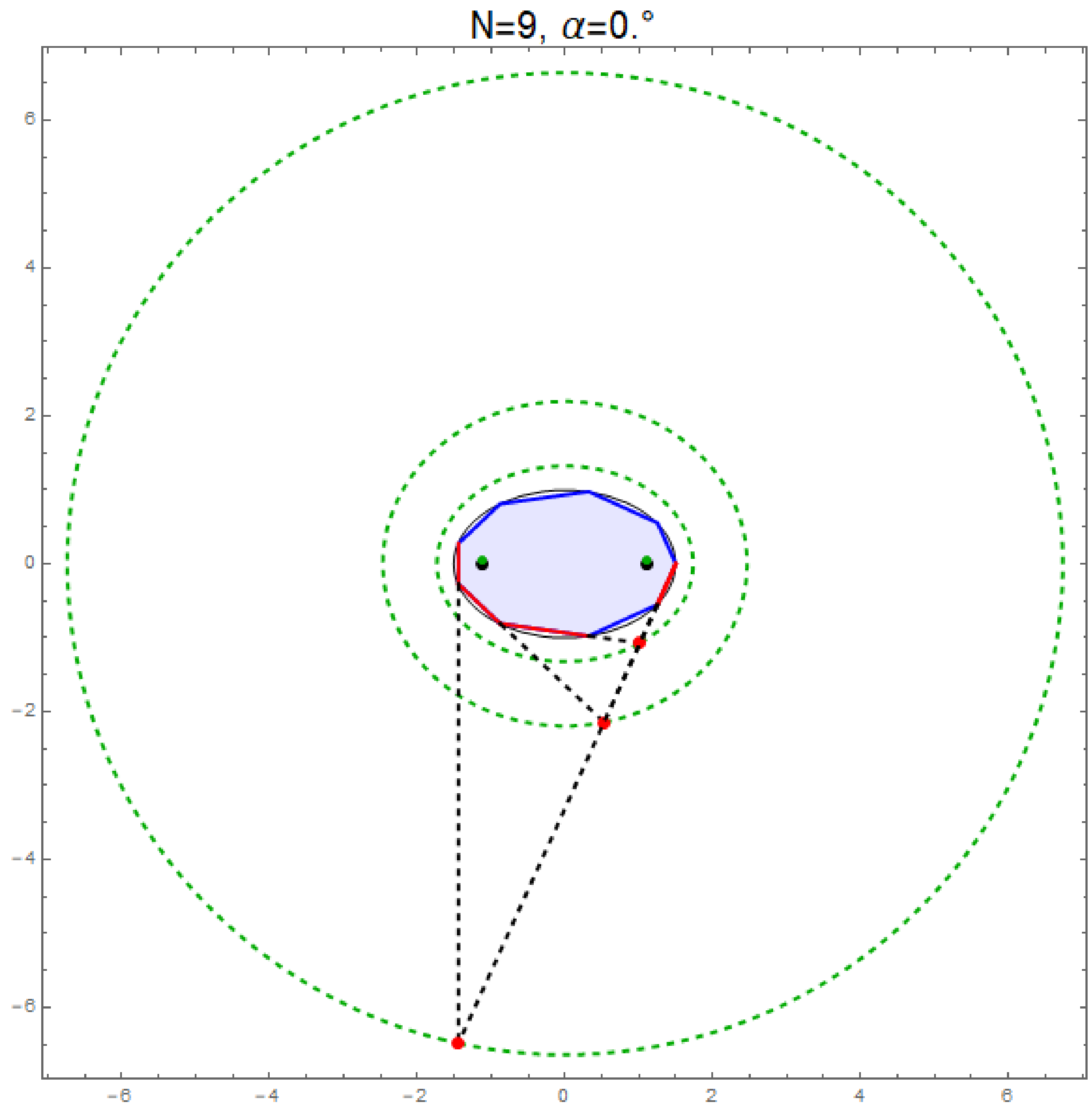
VIDEO

APPENDIX

STATIONARY CIRCLES



CAN YOU GET A CIRCLE FROM AN ELLIPSE?



Jean-Gaston Darboux
1842-1917

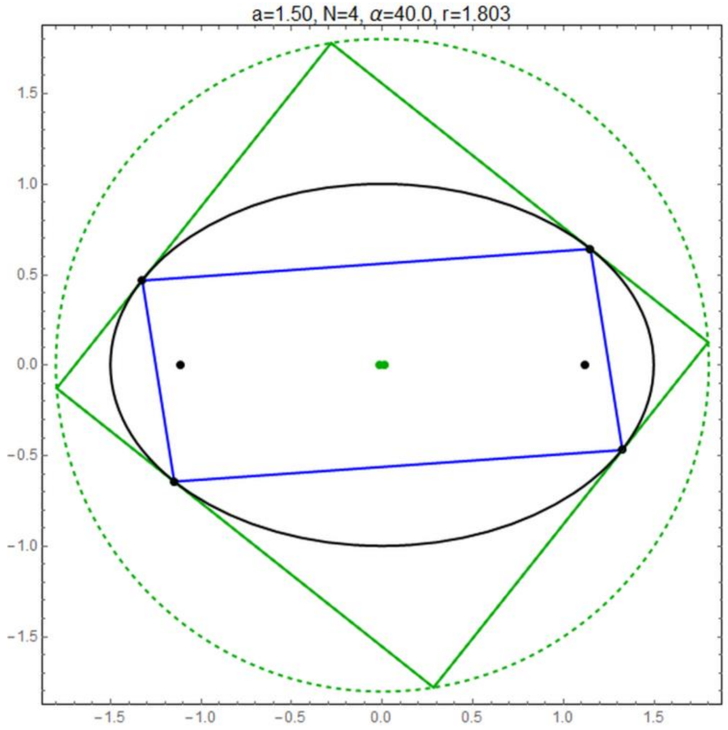
PONCELET-DARBOUX GRID

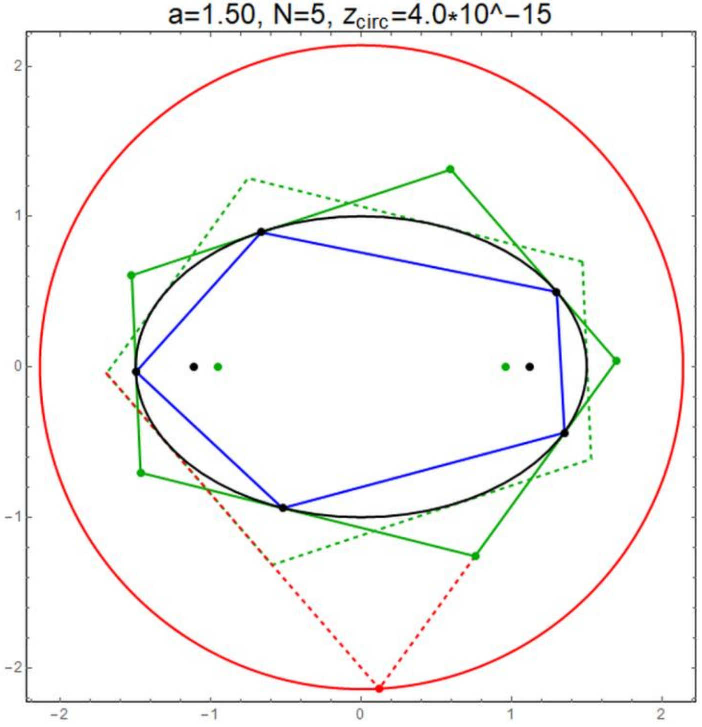
VIDEO

Jean-Gaston Darboux
1842-1917

MONGE-DARBOUX CIRCLES: A COMMON RULE

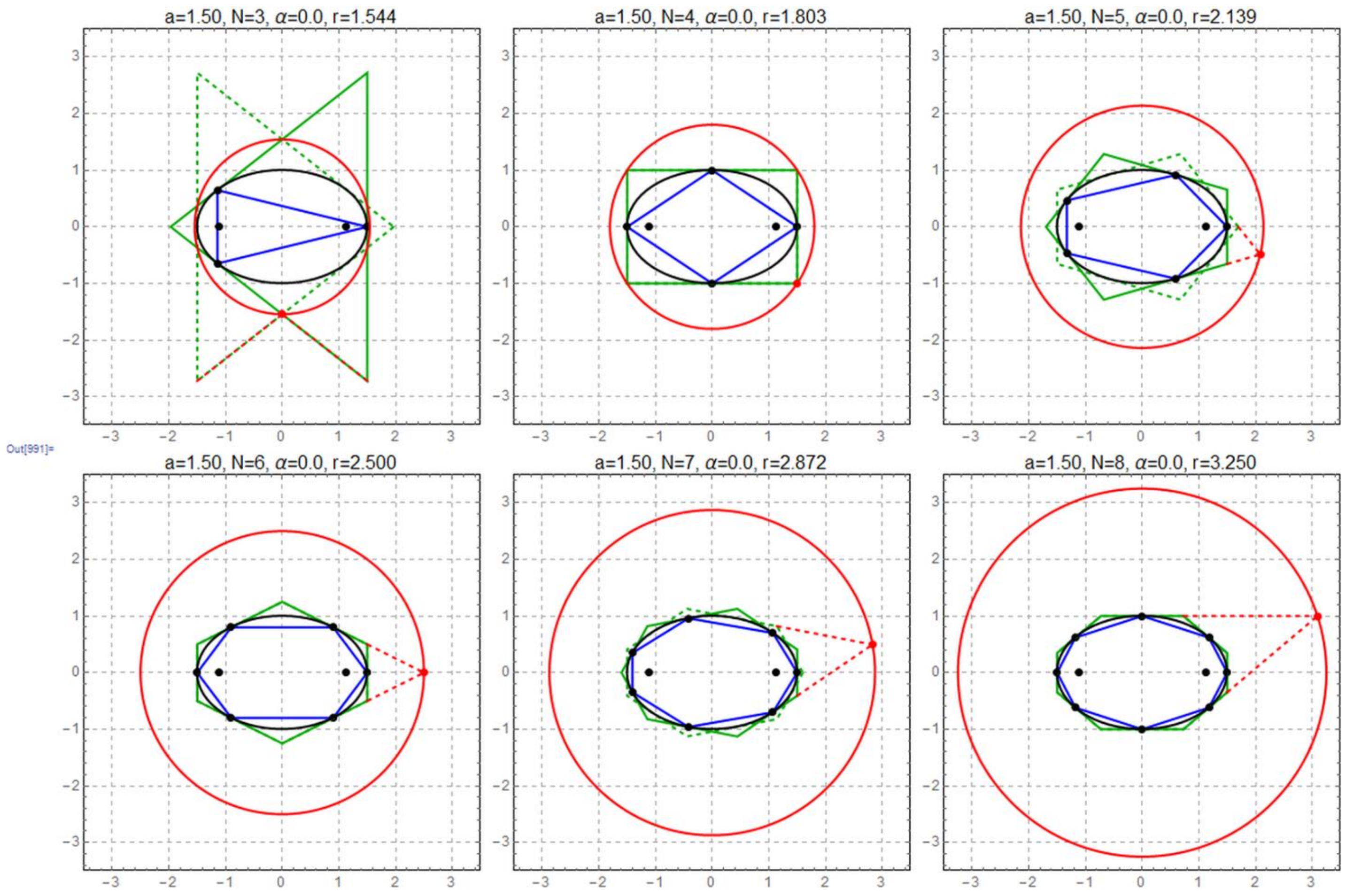
[VIDEO]



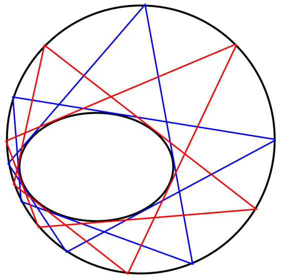


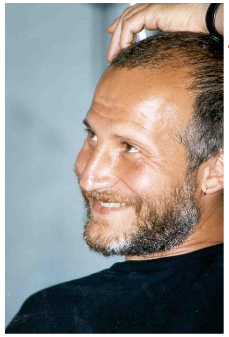
STATIONARY CIRCLES FOR ALL!

VIDEO



This is a very nice fact. I did not know it before. Theorems of this type can be seen by analyzing a projective Poncelet picture. --Arseniy Akopyan, July 13, 2019



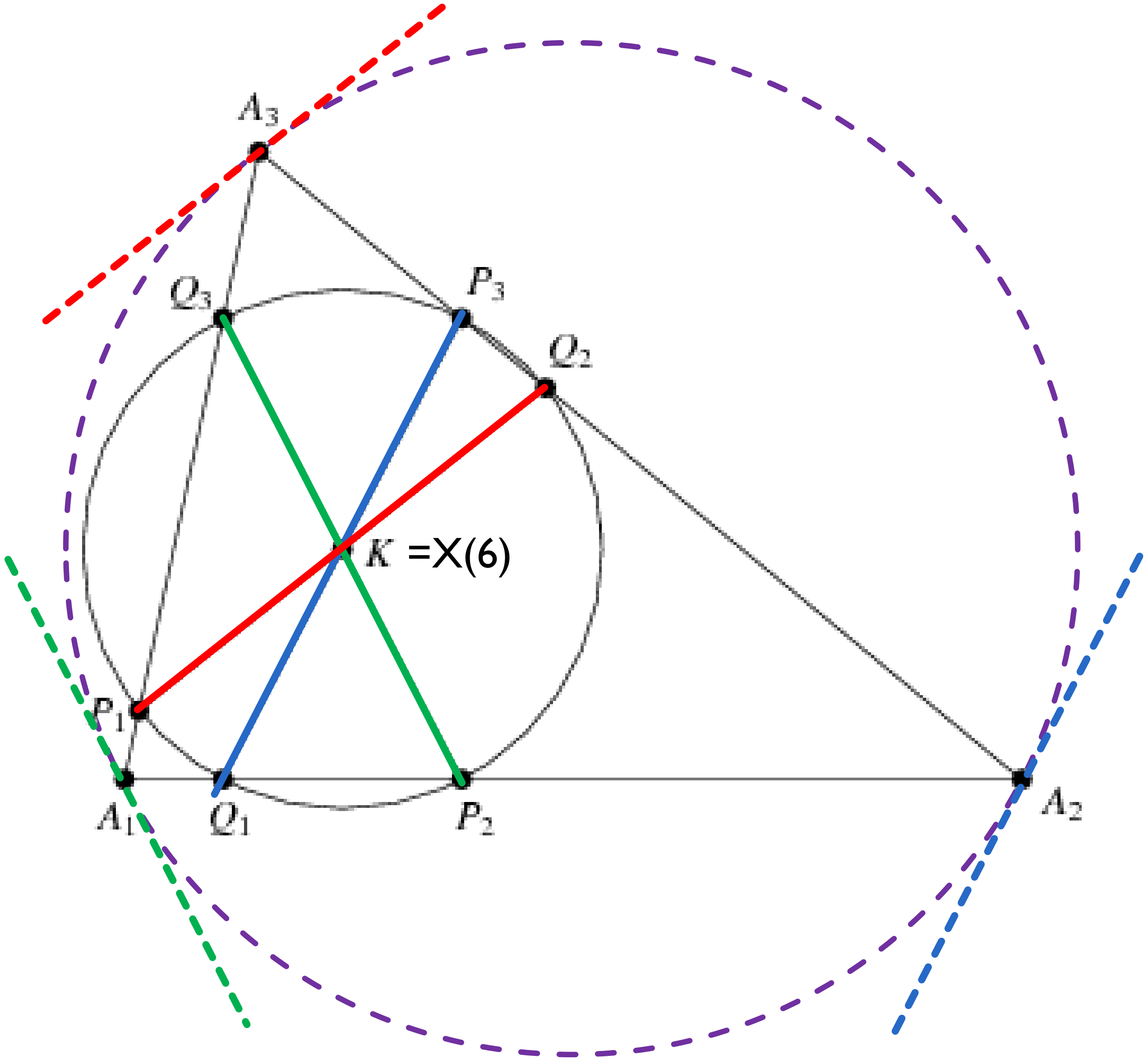


I didn’t know this fact, and I don’t know whether it’s new. However, here is an easy proof. --Sergei, July 13, 2019

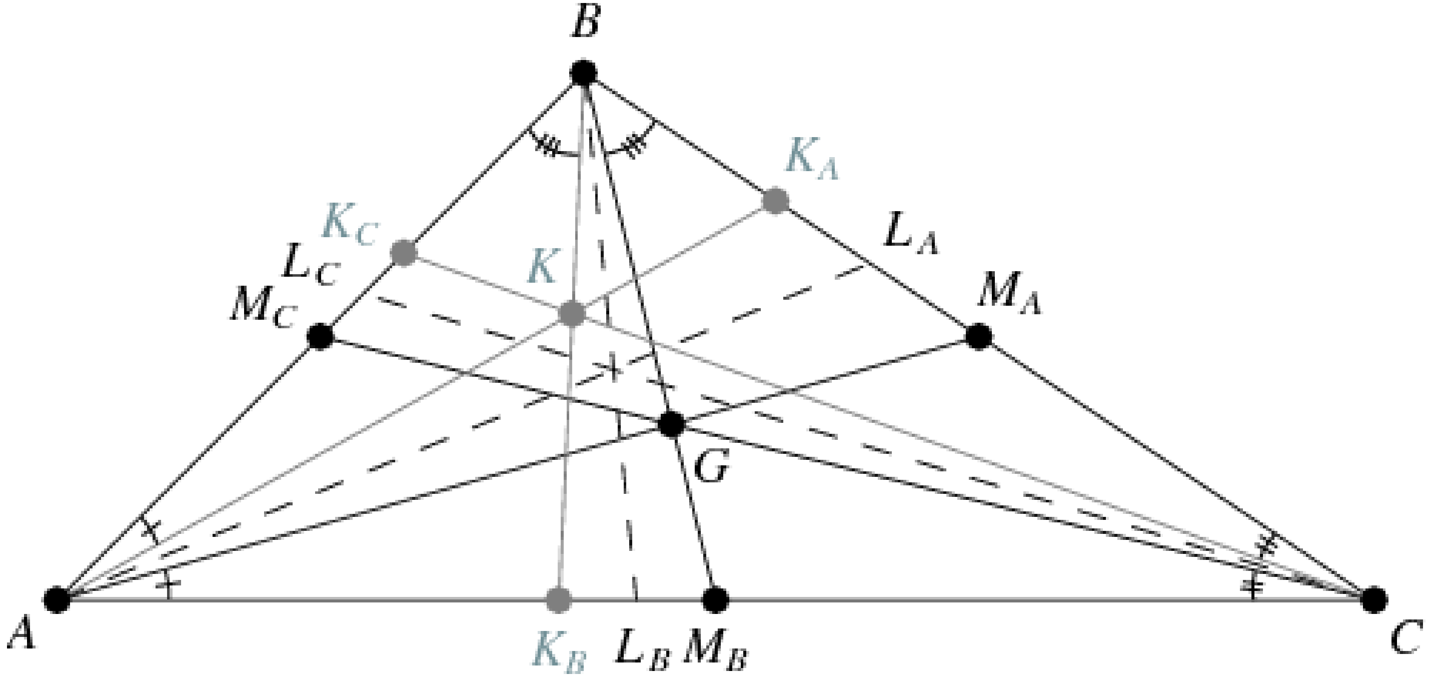
This new phenomenon doesn't seem to follow from the Poncelet Grid [...] I don't really see how to prove this circle result right away. --Richard, July 16, 2019

MEET THE COSINE CIRCLE

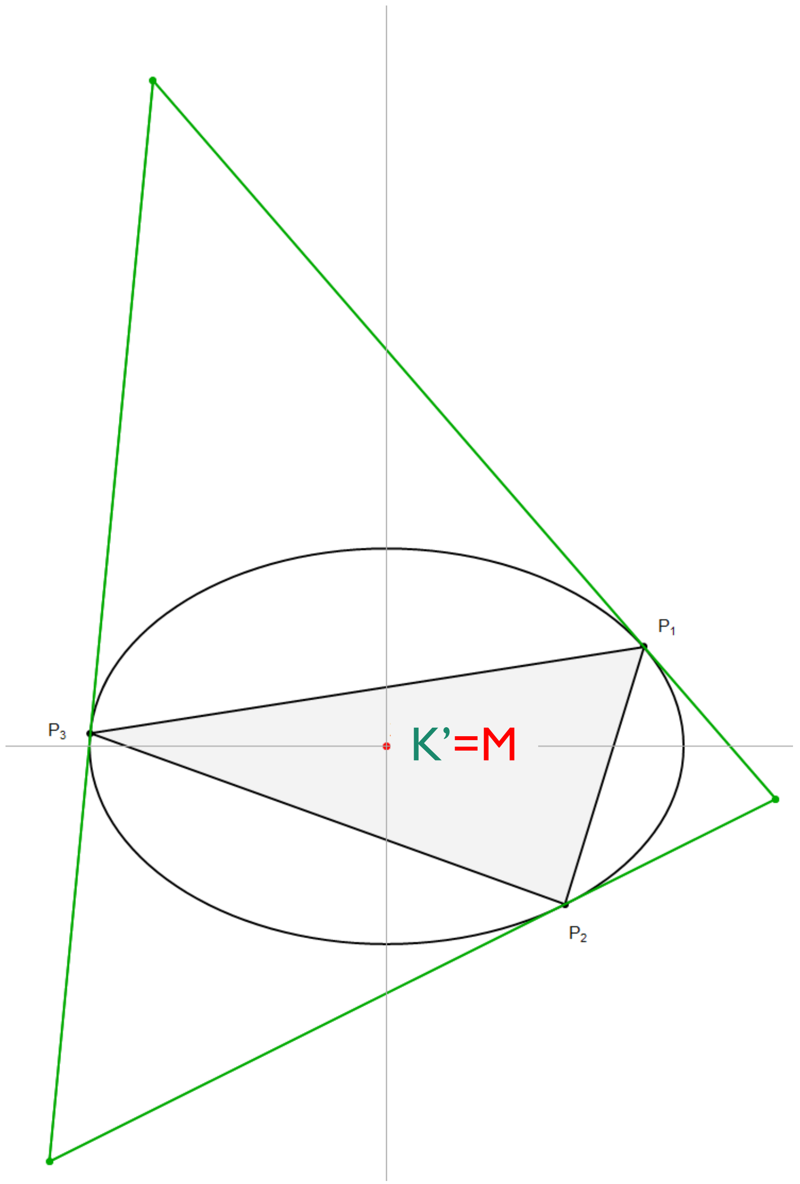
LINK



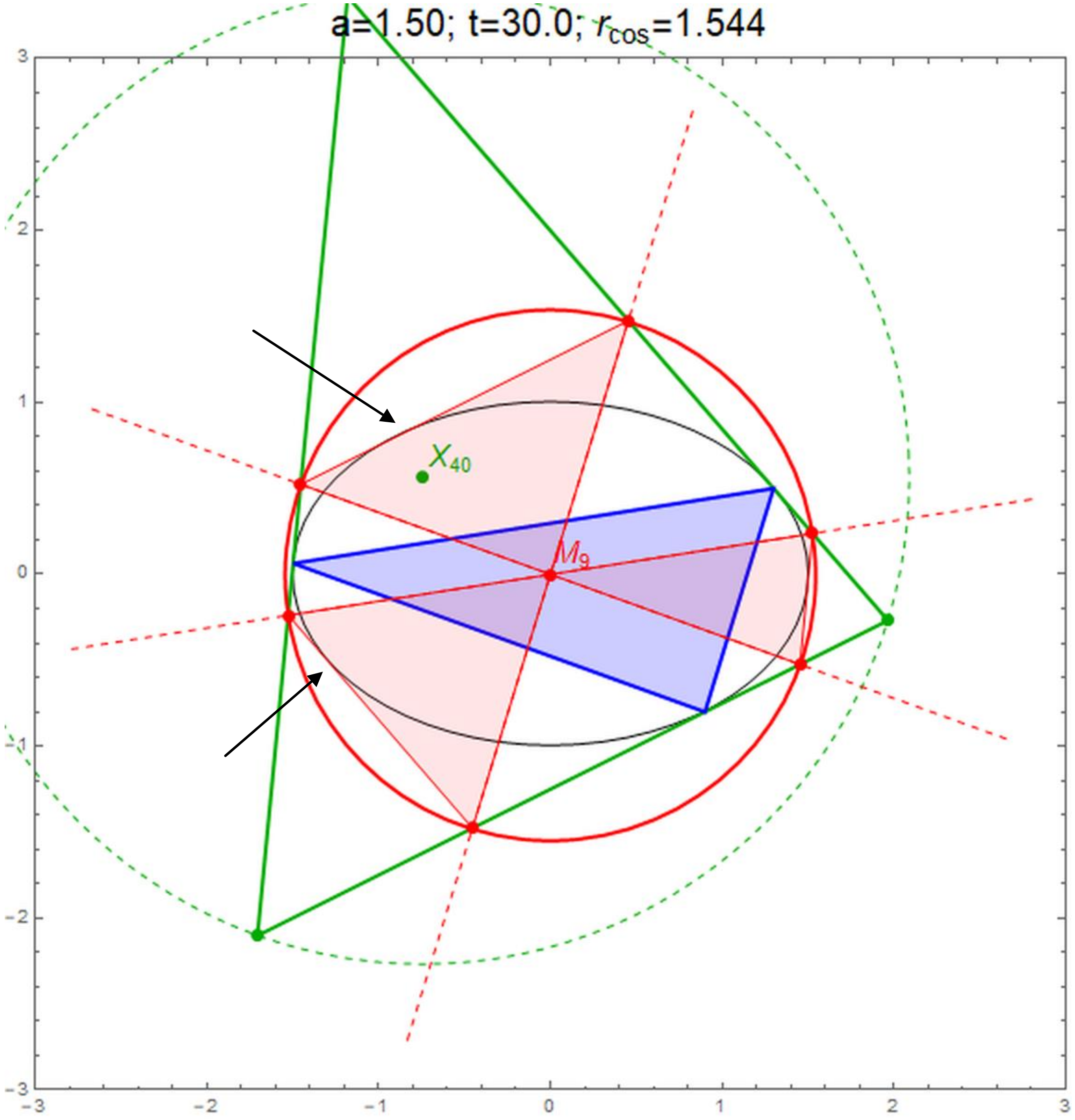
SYMMEDIAN POINT X(6)



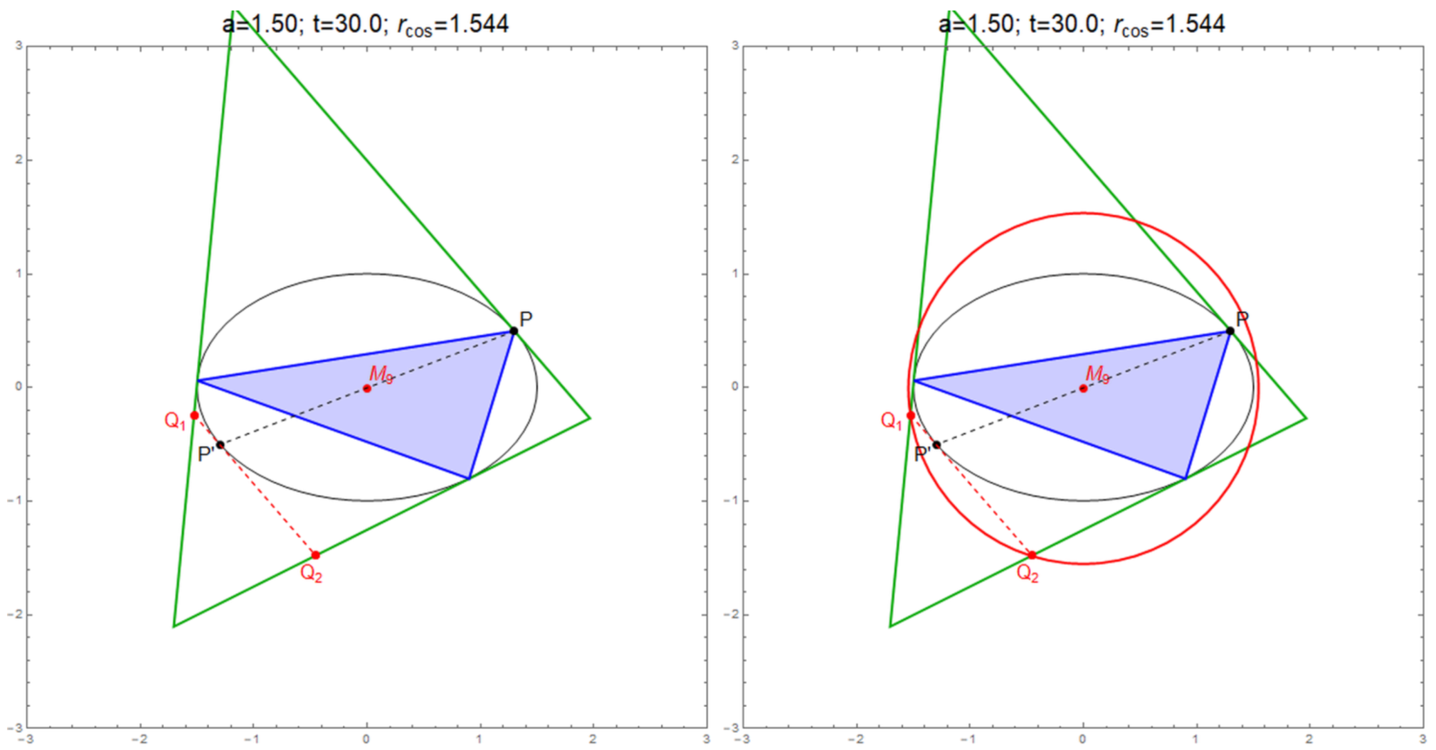
SYMM OF EXCENTRAL = MITTEN OF REFERENCE



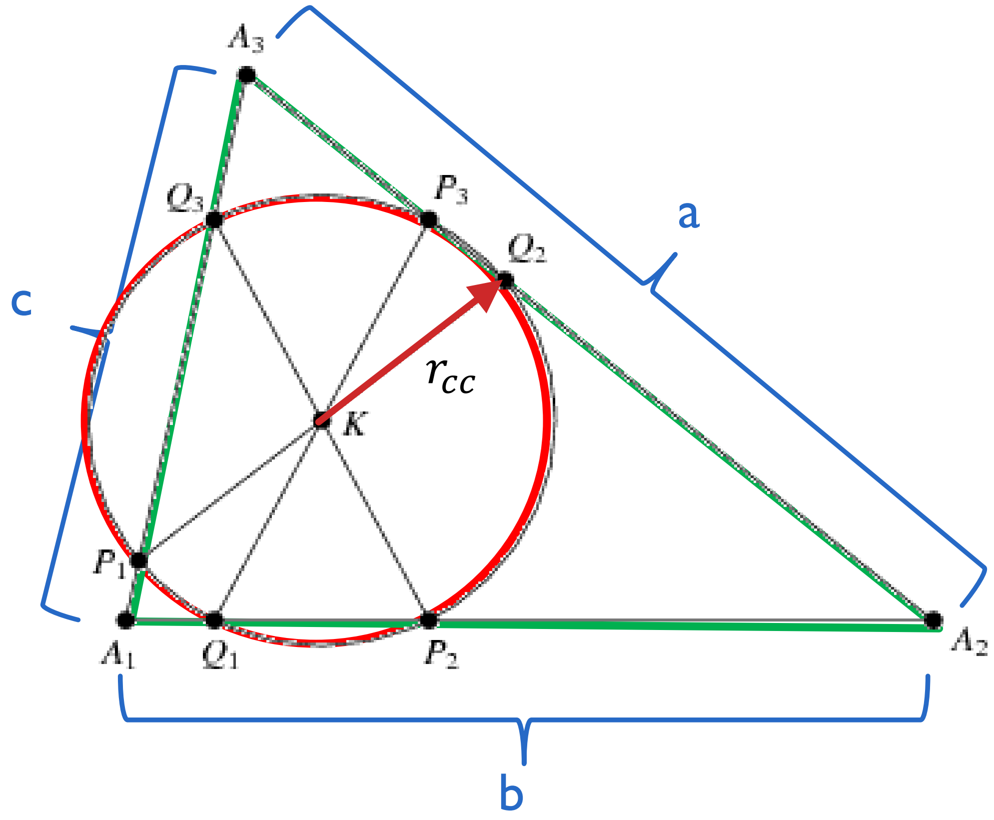
COSINE CIRCLE CONSTRUCTION



SIMPLER COSINE CIRCLE CONSTRUCTION



RADIUS OF COSINE CIRCLE IS NON-LINEAR ON SIDES

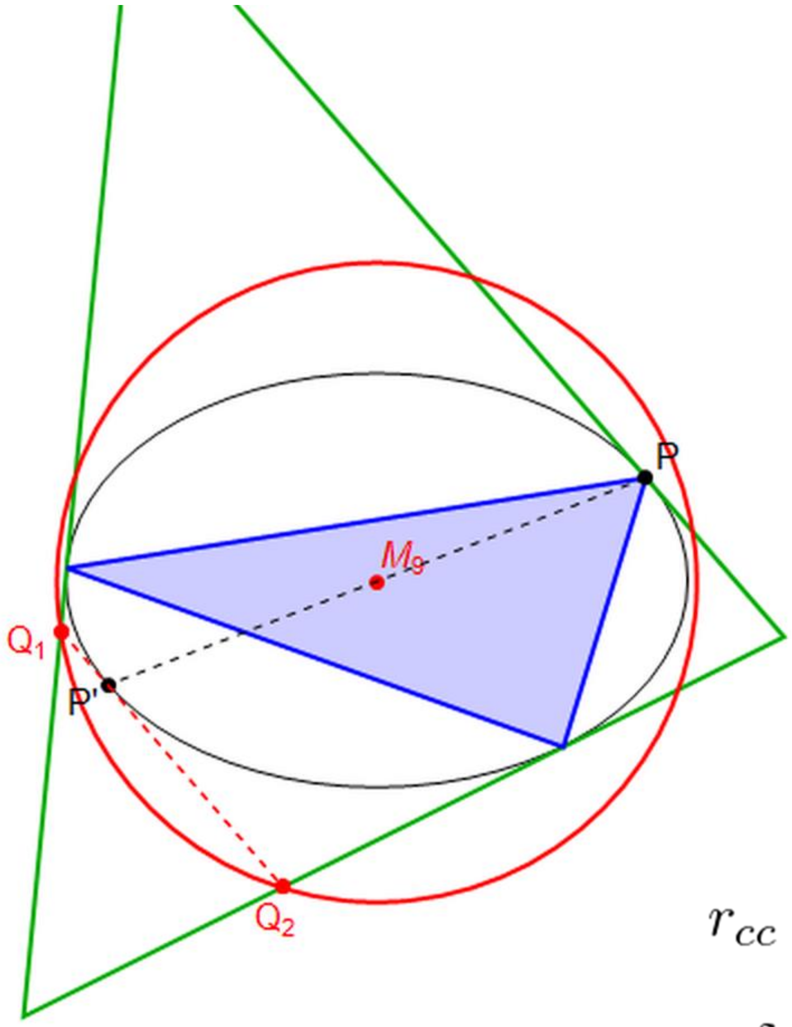


$$r_{cc} = \frac{a\,b\,c}{a^2 + b^2 + c^2}$$

RADIUS REMAINS CONSTANT FOR ALL ORBITS!

VIDEO 1

VIDEO 2



$$r_{cc} = \frac{\sqrt{3}}{3}\sqrt{2\delta + a^2 + 1}$$

$$\delta = \sqrt{a^4 - a^2 + 1}$$

APPENDIX

CONSERVATION EQUATIONS

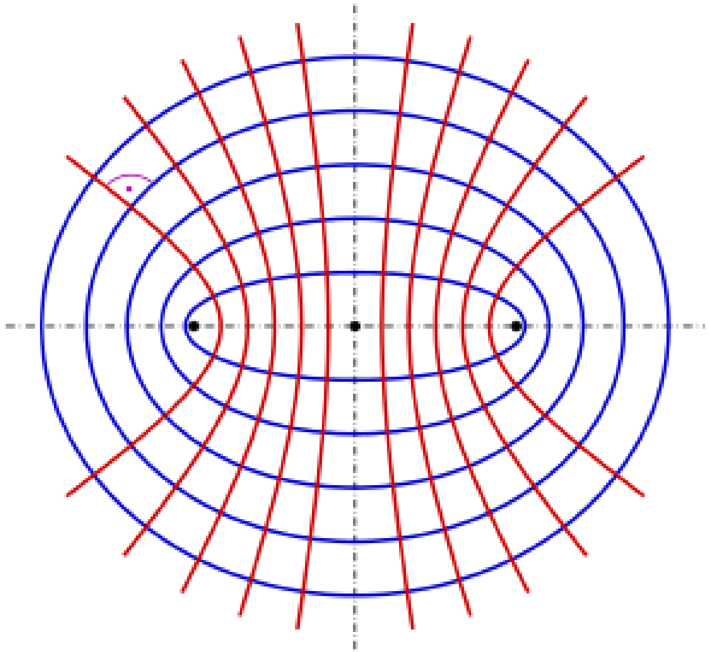
The caustic $\lambda_2 = \kappa$

Jacobi showed in 1838 that the Hamiltonian system for the geodesics in a triaxial ellipsoid can be explicitly integrated. Later, in his Vorlesungen, he used used confocal quadrics coordinates in $n+1$ dimensions. For $n = 2$ one has

$$\frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} + \frac{z^2}{c-\lambda} = 1 \ , \ \ a > \lambda_1 \geq b \geq \lambda_2 \geq c \geq \lambda_3$$

fixing $\lambda_3 = 0$. The method is now called Hamilton-Jacobi's and paved the way for the theory of integrable hamiltonian systems. One verifies first that the kinetic energy kind of separates in variables λ_1, λ_2 . Then the generalized velocities $\dot{\lambda}_1$, $\dot{\lambda}_2$ can be integrated by quadratures.

The elliptic billiard then results from making the smaller axis $c \to 0$. One obtains a double faced elliptical region, parametrized by confocal conics given by $\lambda_1 =$ const (hiperbolae) and $\lambda_2 =$ const (ellipses). The foci are at $\lambda_1 = \lambda_2 = b$, so $x = \sqrt{a-b}$, $y = 0$. The billiard is the outer ellipse $\lambda_2 = 0$.



When one successfully applies the HJ method, a generating function for a suitable canonical transformation is produced, so that besides the value of the energy, additional $n-1$ integrals of motion pop out. In the elliptical billiard one recognizes the second integral as the product $M = M_1 M_2$ of the angular momenta with respect to the foci.

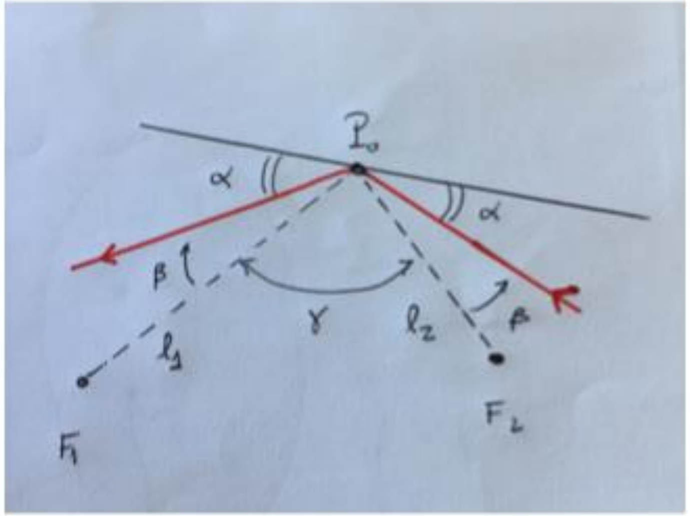
There are two regimes according to the sign of M.

$M = 0$ corresponds to the trajectories passing through the foci. Consider for instance the regime $M > 0$. Defining $\kappa = b - M/2h$ (see below), a closer look at the generating function shows that $0 \leq \lambda_2 \leq \kappa < b$. This means that that the projection of the trajectories in phase space to the configuration space are confined to the annular region between the ellipses corresponding to $\lambda_2 = 0$, and $\lambda_2 = \kappa$. Thus the inner ellipse

$$\frac{x^2}{a-\kappa} + \frac{y^2}{b-\kappa} = 1$$

is a caustic curve for the projections of the (Lagrangian) torus corresponding to fixing the values of h and M.

Joachimsthal: $M_1 M_2$ is conserved



Before the collision:

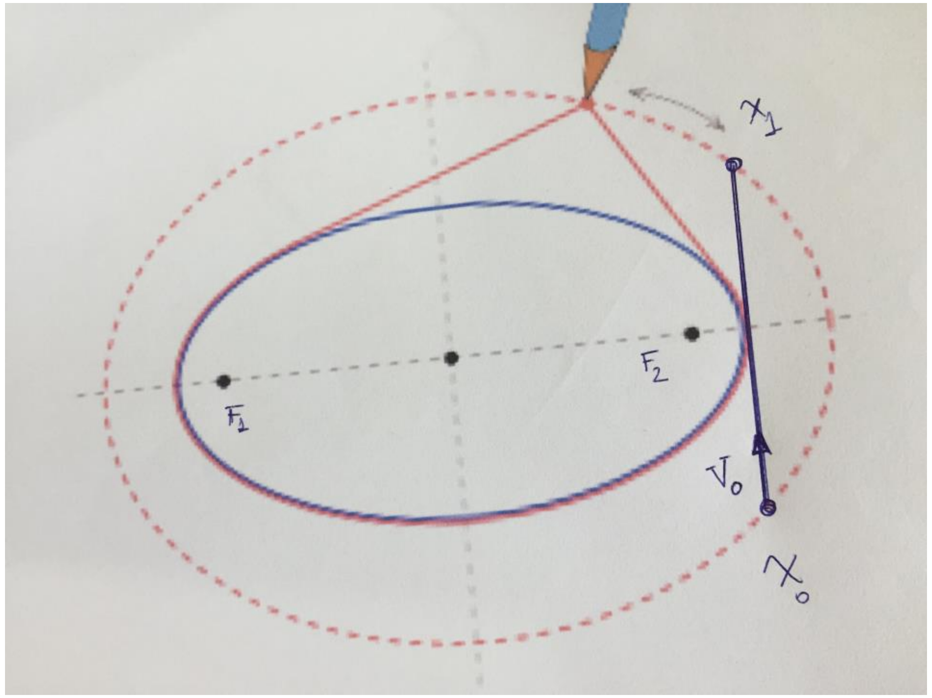
$$M_1 = \ell_1 \sin(\pi - \gamma - \beta) = \ell_1 \sin(\gamma + \beta)\ ,\ M_2 = \ell_2 \sin\beta .$$

After the collision:

$$M_2 = \ell_2 \sin(\pi - \gamma - \beta) = \ell_2 \sin\gamma + \beta)\ ,\ M_1 = \ell_1 \sin\beta .$$

$$\frac{x^2}{a-\kappa}+\frac{y^2}{b-\kappa}=1$$

$$F_i=(\pm\sqrt{a-b},0)$$



To prove these relationships, just get the coordinates of X_o

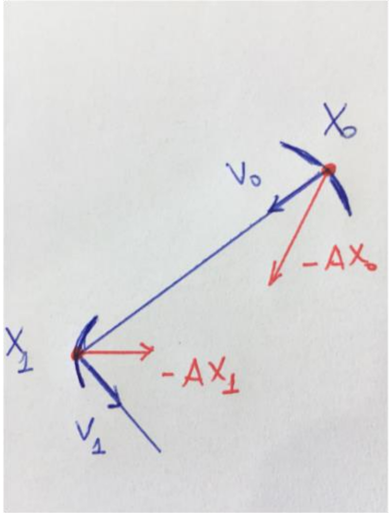
$$c=v\sqrt{\kappa/ab}\,.$$

$$M=M_1M_2=v^2(b-\kappa)$$

The "stroboscopic" integral $(-AX, V) = c$

Let $A = \text{diag}(1/a, 1/b), a \geq b > 0$, so the ellipse is given by

$$(AX, X) = 1, \text{ i.e } x^2/a + y^2/b = 1 \ .$$



Take two consecutive bounces in the elliptical billiard

$$X_i = (x_i, y_i), i = 0, 1.$$

The respective outgoing unit vectors are

$$V_o = \frac{X_1 - X_o}{|X_1 - X_o|}, \ V_1 = V_o - 2\alpha \hat{N}_1$$

$$\hat{N}_1 = \frac{AX_1}{|AX_1|} = \frac{1}{\sqrt{x_1^2/a^2 + y_1^2/b^2}} (x_1/a, y_1/b), \quad \alpha = \left(V_o \, , \, \hat{N}_1\right)$$

$$(AX_o, V_o) \stackrel{?}{=} (AX_1, V_1)$$

The left hand side is

$$c_o = (AX_o, V_o) = d\ (x_o(x_1 - x_o)/a, y_o(y_1 - y_o)/b) = d(x_0\, x_1/a + y_o\, y_1/b - 1)$$

where

$$d = \frac{1}{\sqrt{(x_1 - x_o)^2 + (y_1 - y_o)^2}}.$$

Let us compute the right hand side, given by

$$c_1 = (x_1/a, y_1/b) \cdot \left(V_o - 2\alpha\, \hat{N}_1\right).$$

A quick calculation yields

$$c_1/d = [x_1(x_1 - x_o)/a + y_1(y_1 - y_o)/b] - 2\,[x_1(x_1 - x_o)/a + y_1(y_1 - y_o)/b]$$

Taking into account that $x_o^2/a + y_o^2/b = x_o^2/a + y_o^2/b = 1$,
we observe that the multiplicative factor 2 makes the +1 to become -1 and the product $x_o x_1 - y_o y_1$ to change sign.

$$\boxed{c = (1 - X_o A X_1)/|X_1 - X_o|}$$

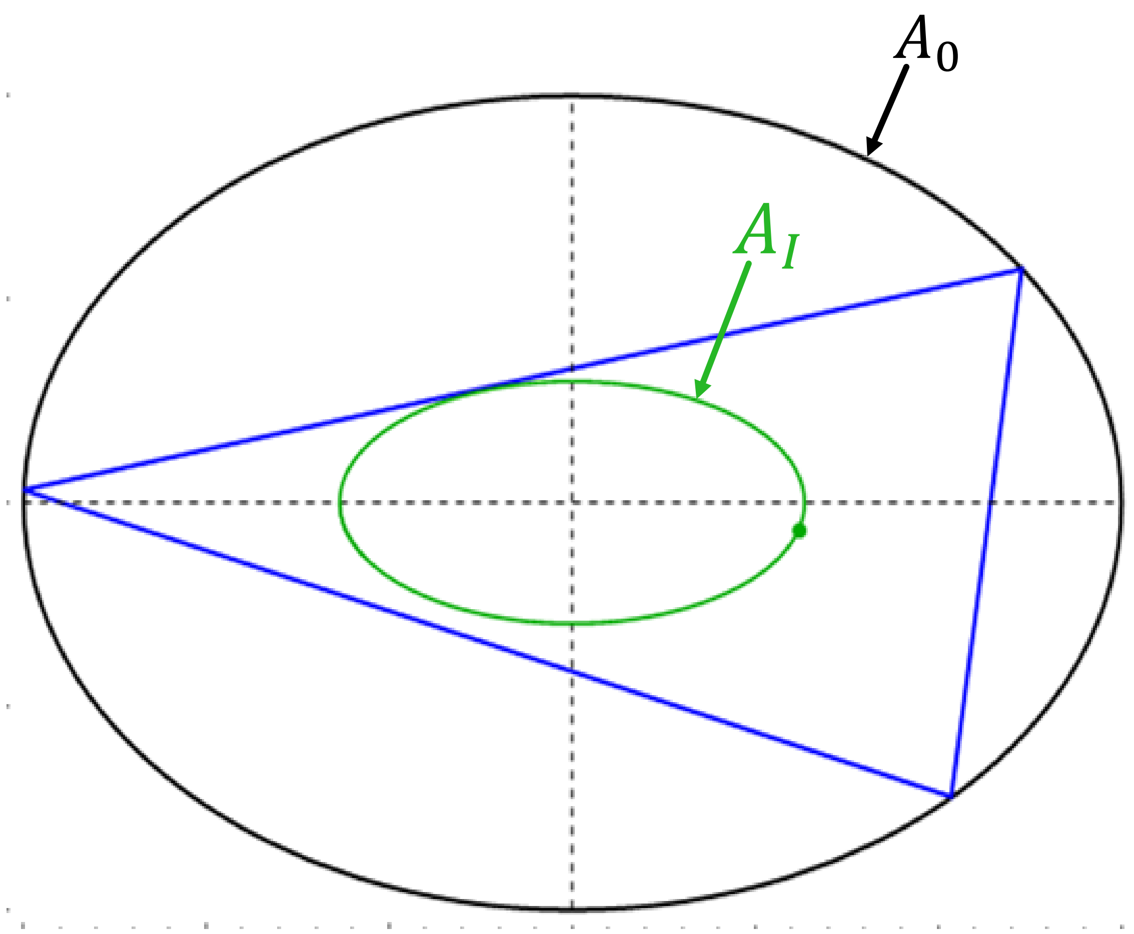
OVERFLOW

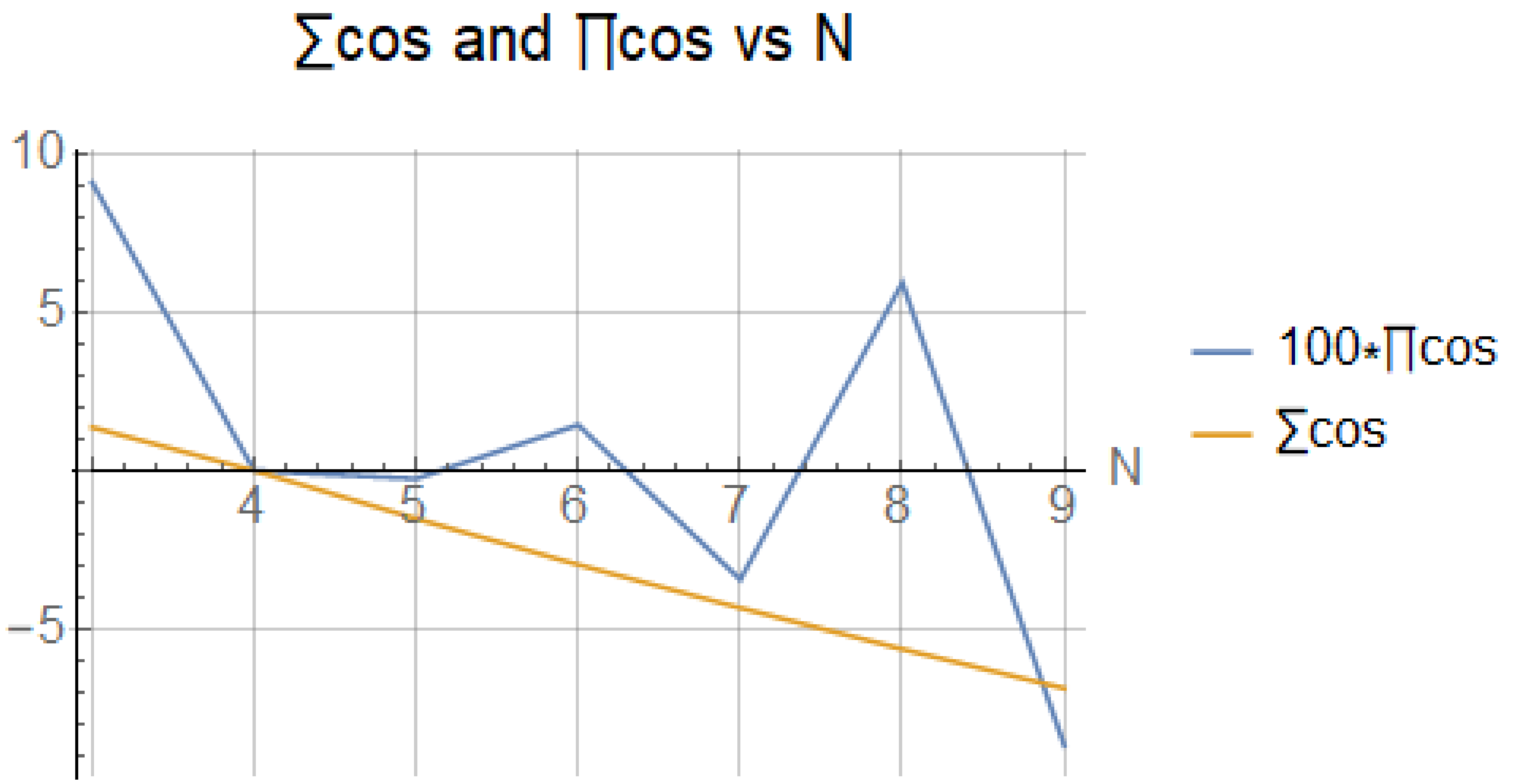
SLIDES NOT USED

					
Gaspard **Monge** 1746–1818	Jean-Victor **Poncelet** 1788–1867	Karl Wilhelm **Feuerbach** 1800–1834	Christian Heinrich **von Nagel** 1803–1882	Ferdinand **Joachimsthal** 1818–1861	Jean-Gaston **Darboux** 1842–1917

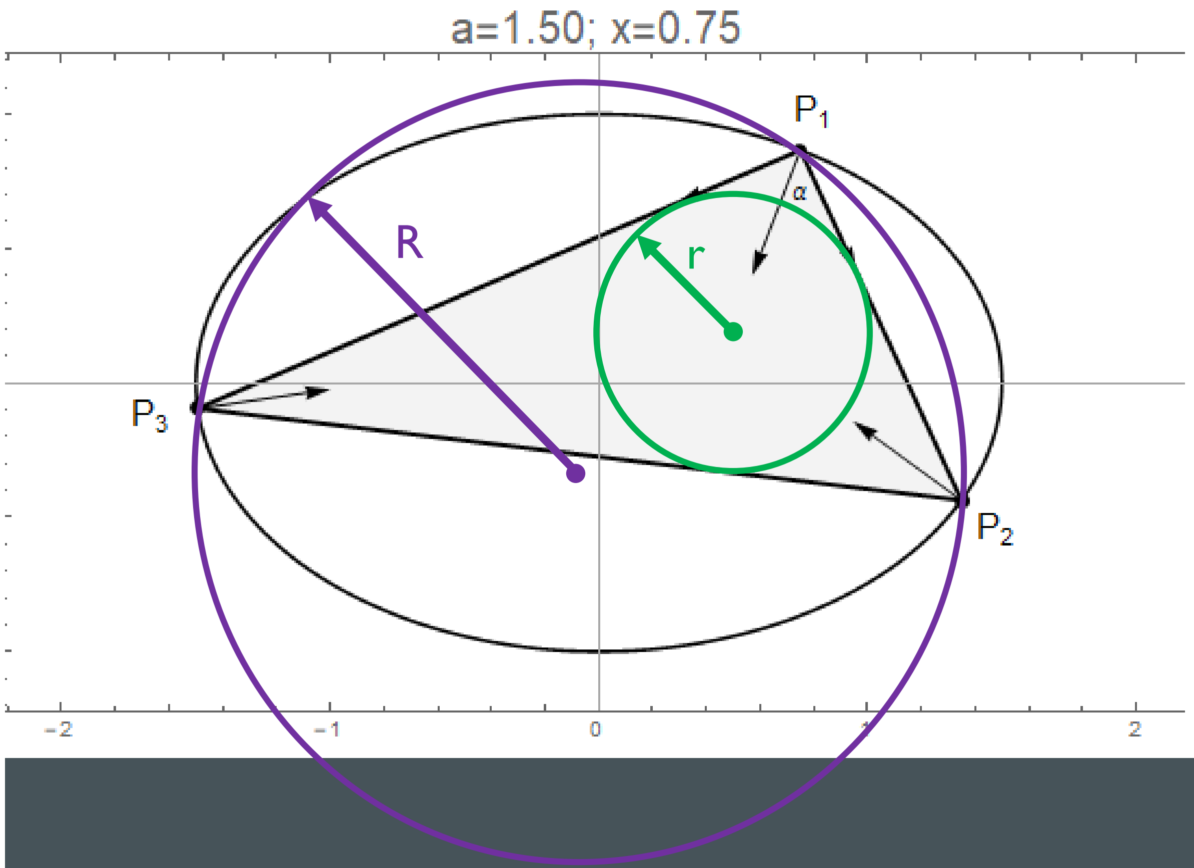
PROOF OF $\frac{r}{R}$ CONSTANCY

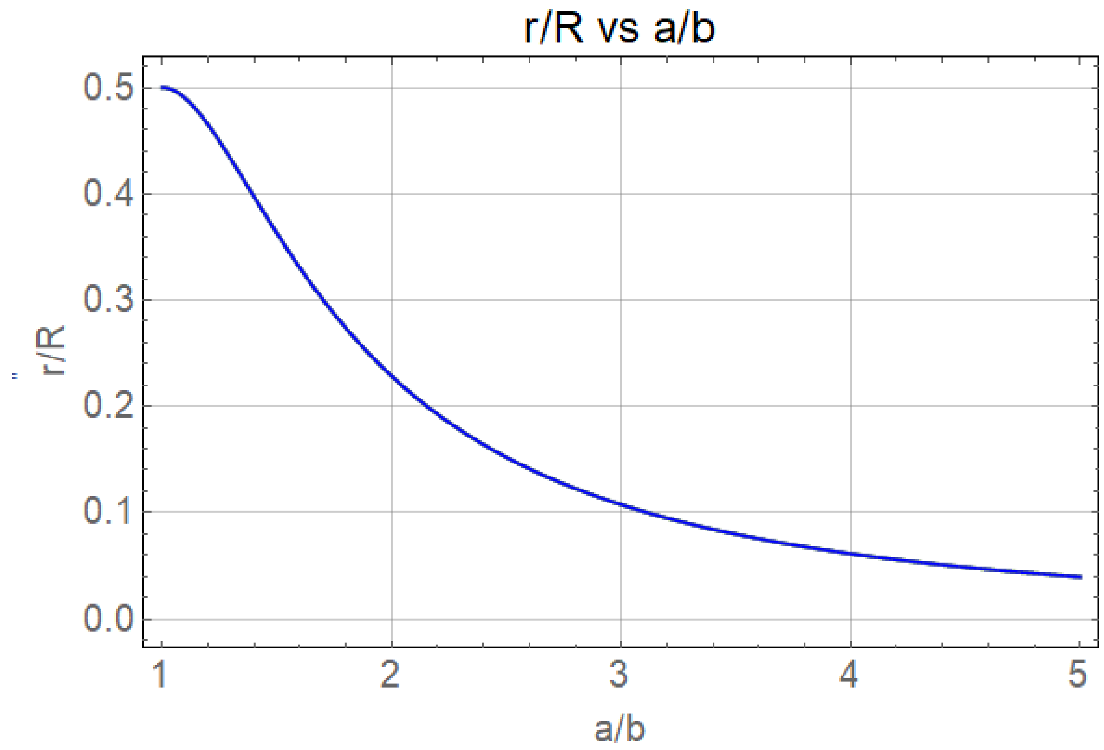
- Incenter locus is elliptic [Prof. Romaskevich]
- Area ratio: $\frac{A_I}{A_0} = \frac{r}{2R}$ [Prof. Garcia]
- Both are stationary => r/R constant.





∑cos and ∏cos vs N
10
5
−5
4
5
6
7
8
9
N
100*∏cos
∑cos

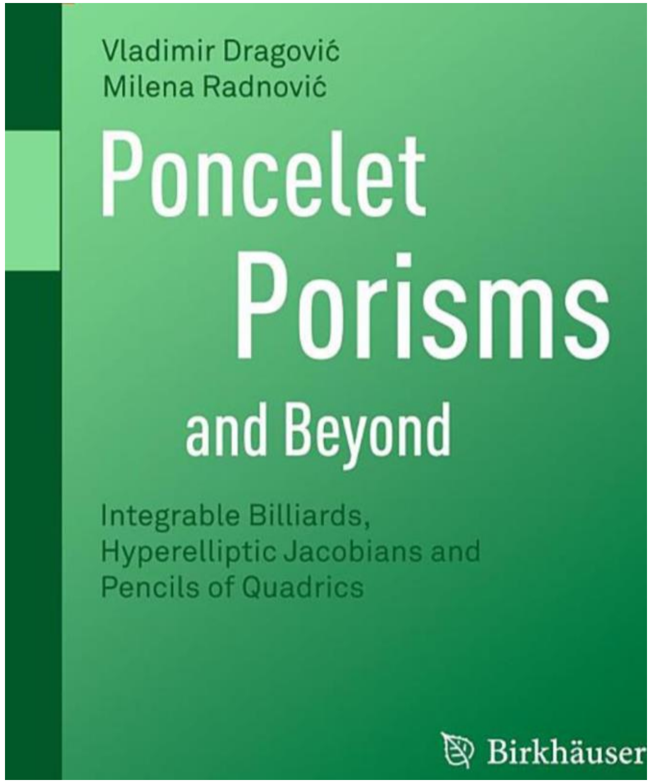




CONSTANT INRADIUS/CIRCUMRADIUS

"Nature uses only the longest threads to weave her patterns, so each small piece of her fabric reveals the organization of the entire tapestry."

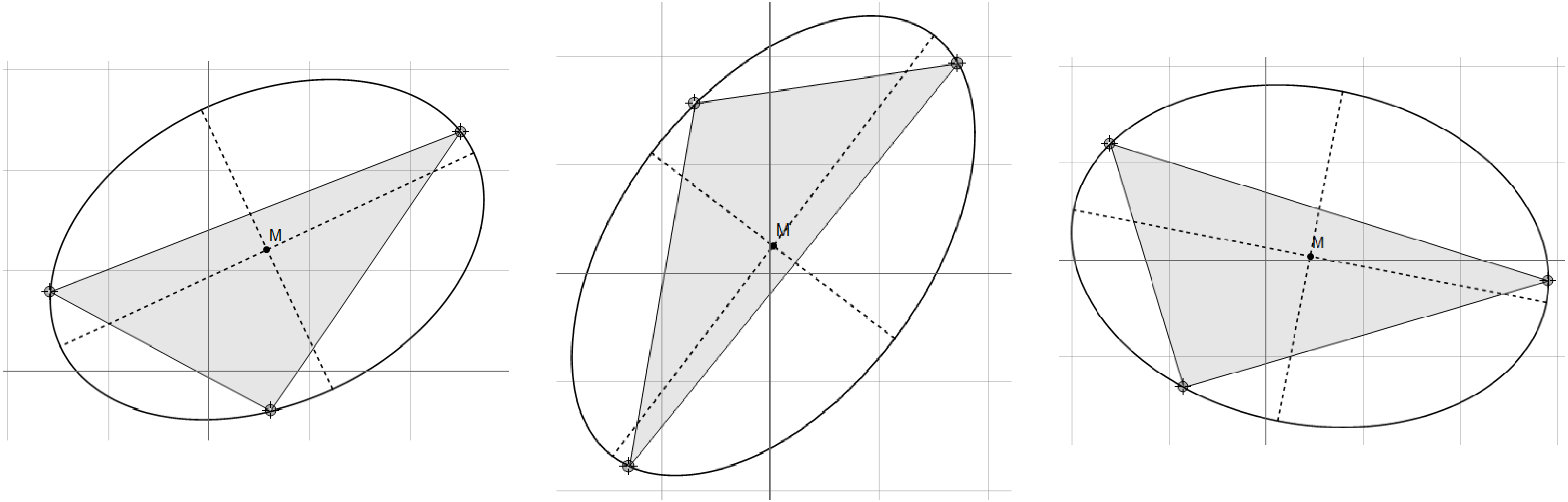
– Richard P. Feynman



Chapter 3

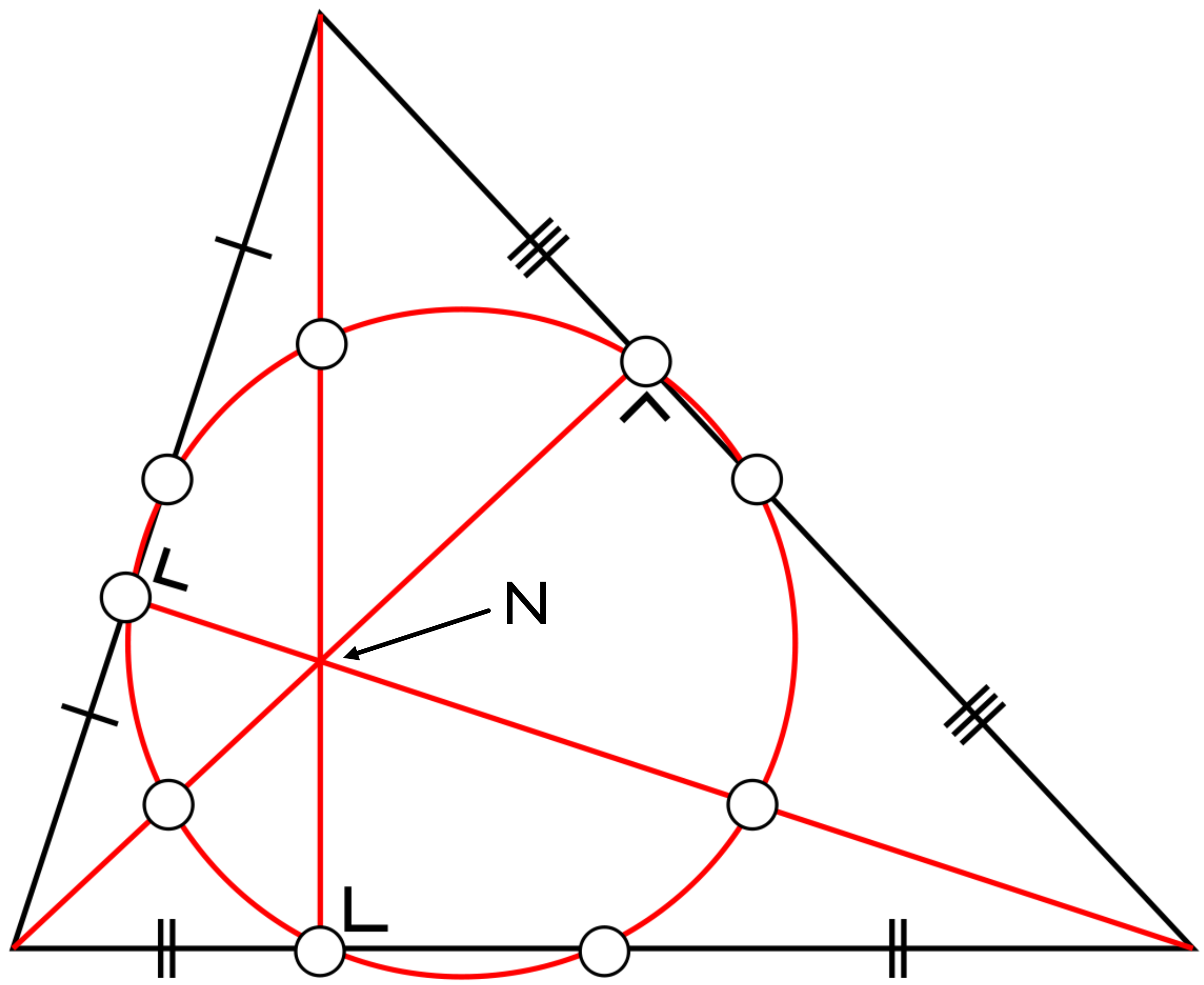
Hyperelliptic Curves and Their Jacobians

Exercise 3.7. Let f be a meromorphic non-constant function on a compact Riemann surface. Prove that, for each $z \in \mathbf{C} \cup \{\infty\}$, the set $f^{-1}(z)$ is not empty.



CIRCUMBILLIARDS

VIDEO AND APPLET



NINE-POINT CIRCLE X(5)

Wolfram MathWorld™
Built with Mathematica Technology

Calculus and Analysis > Dynamical Systems >

Integral of Motion

A function of the coordinates which is constant along a trajectory in phase space. The number of degrees of freedom of a dynamical system such as the Duffing differential equation can be decreased by one if an integral of motion can be found. In general, it is very difficult to discover integrals of motion.

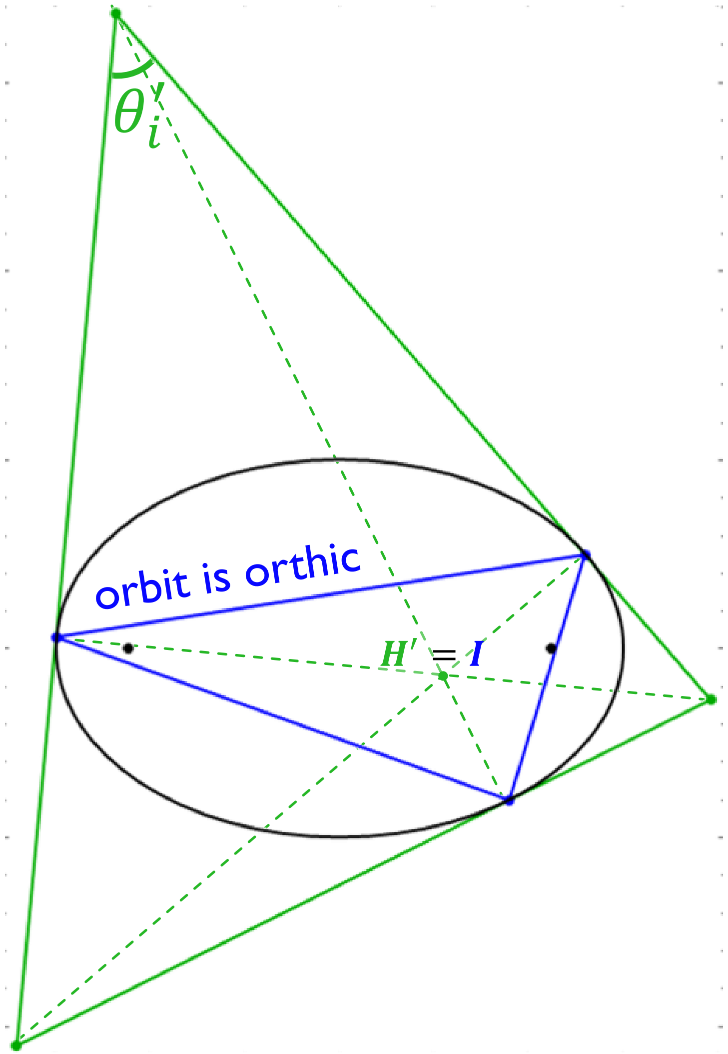
CITE THIS AS:

Weisstein, Eric W. "Integral of Motion." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/IntegralofMotion.html

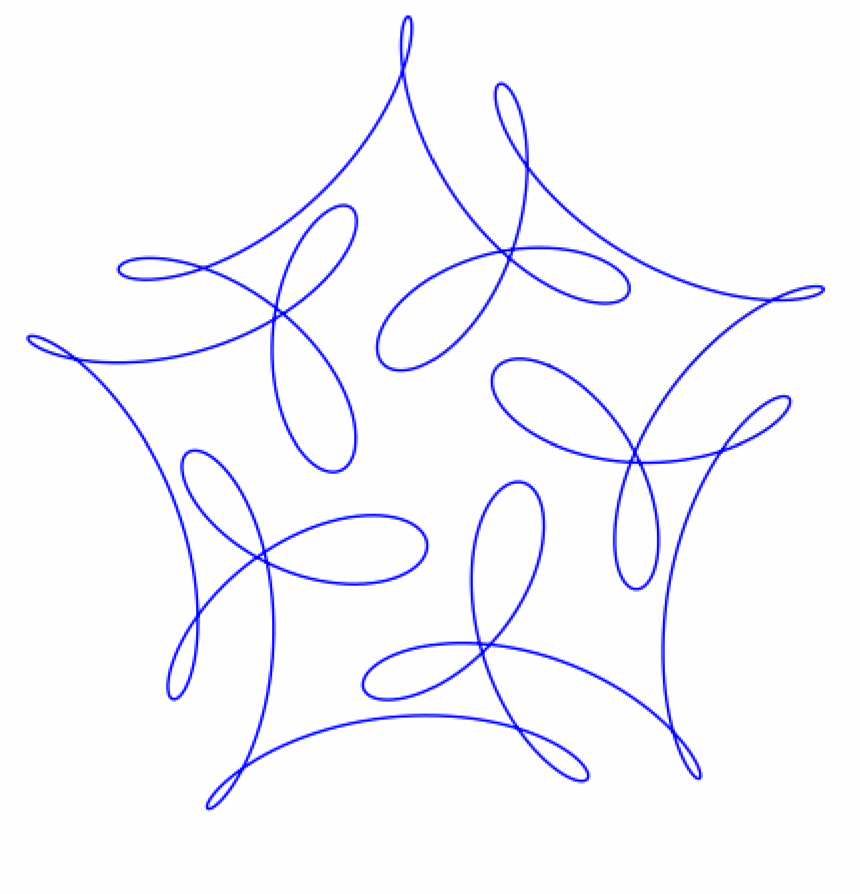
WONDER FORMULA II:

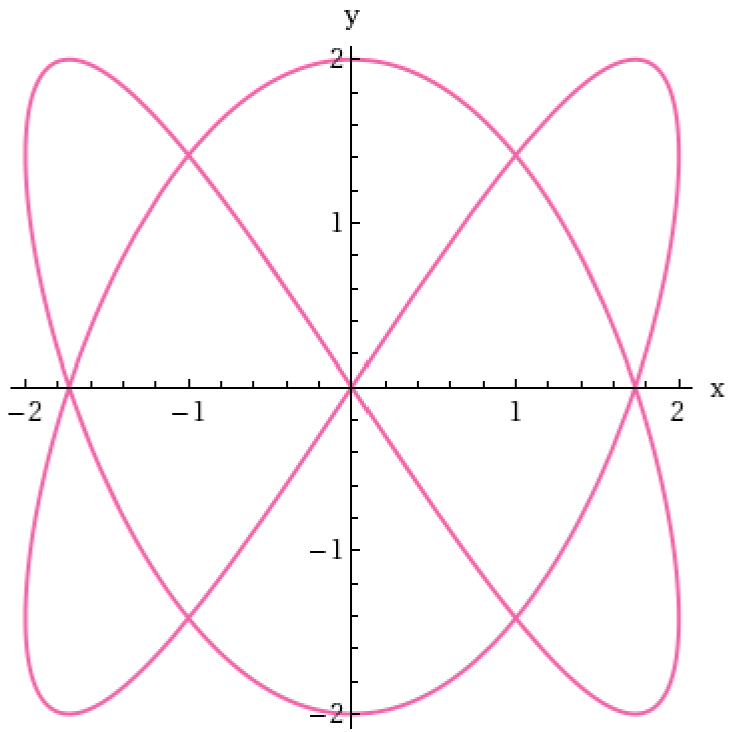
orthic

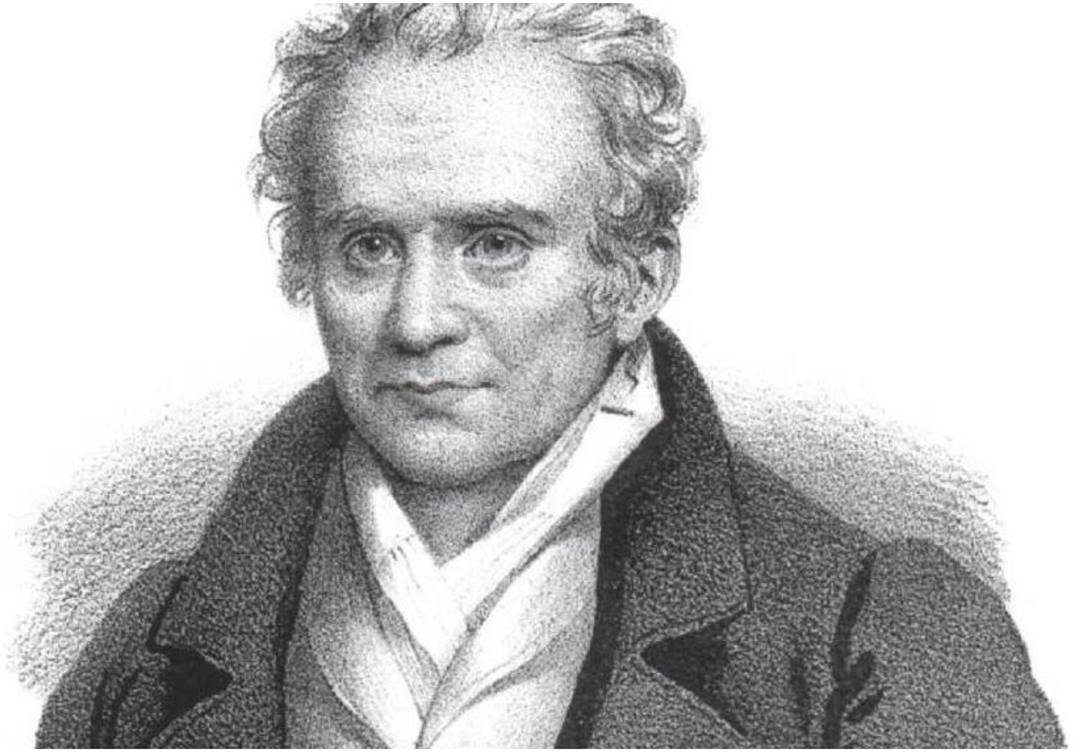
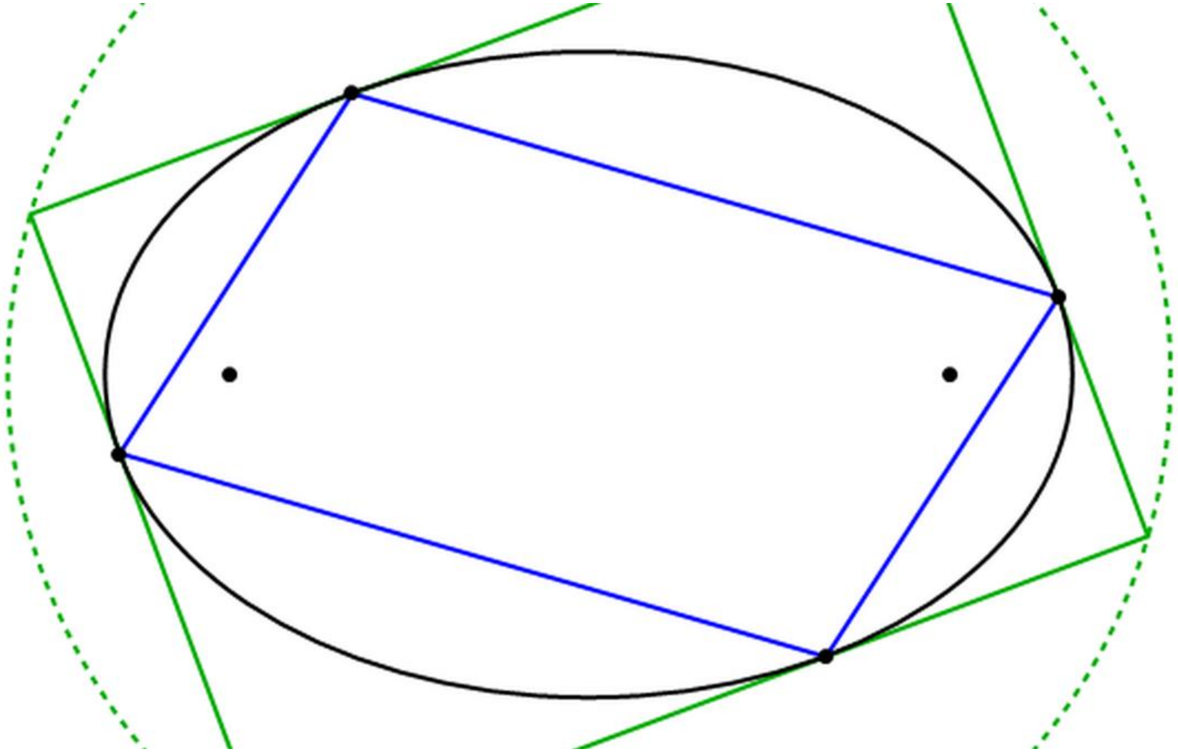
$$\left|\cos\theta_1' \cos\theta_2' \cos\theta_3'\right| = \boxed{\frac{r_H}{4R_H}}$$



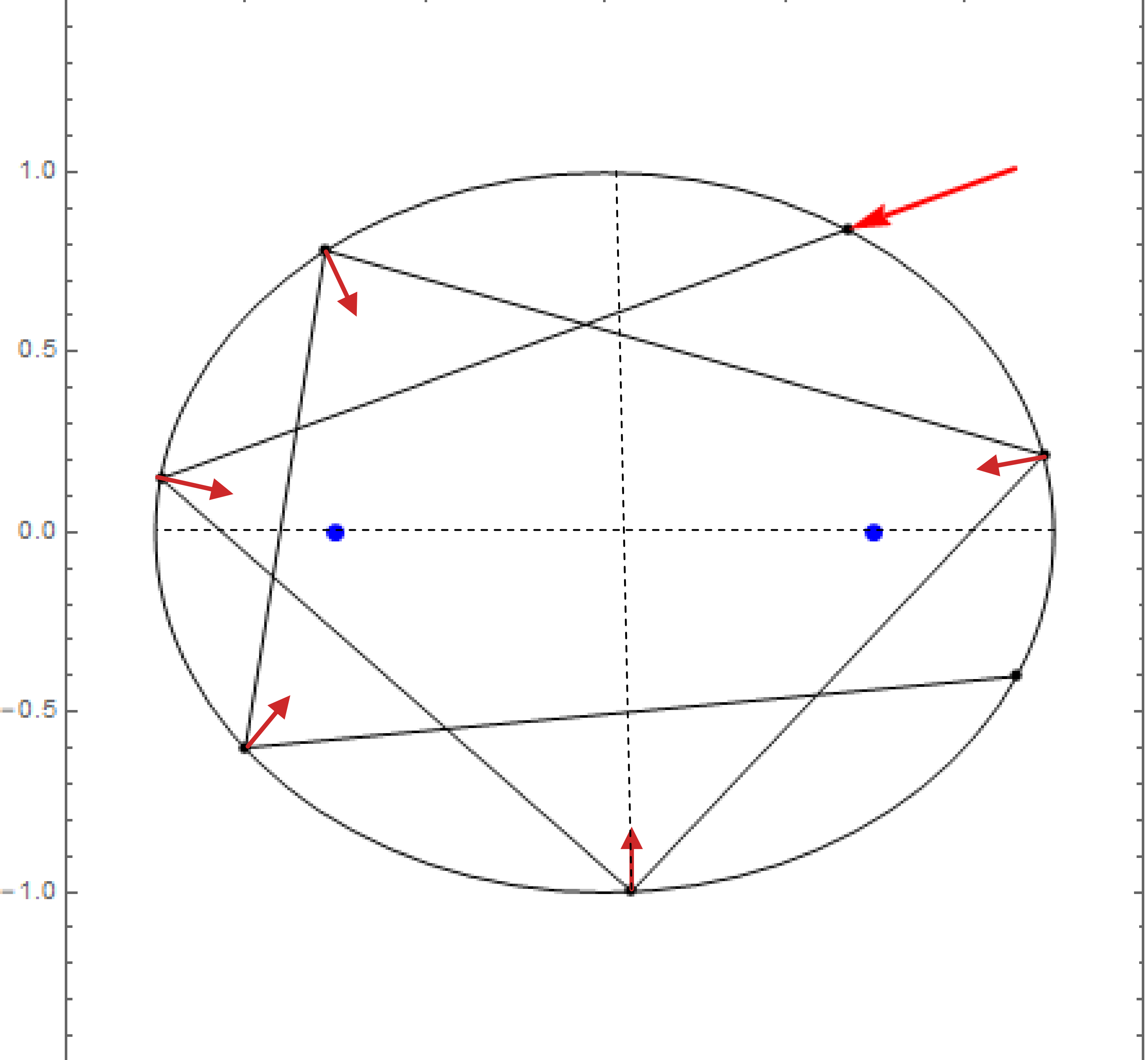
LOCUS / LOCI





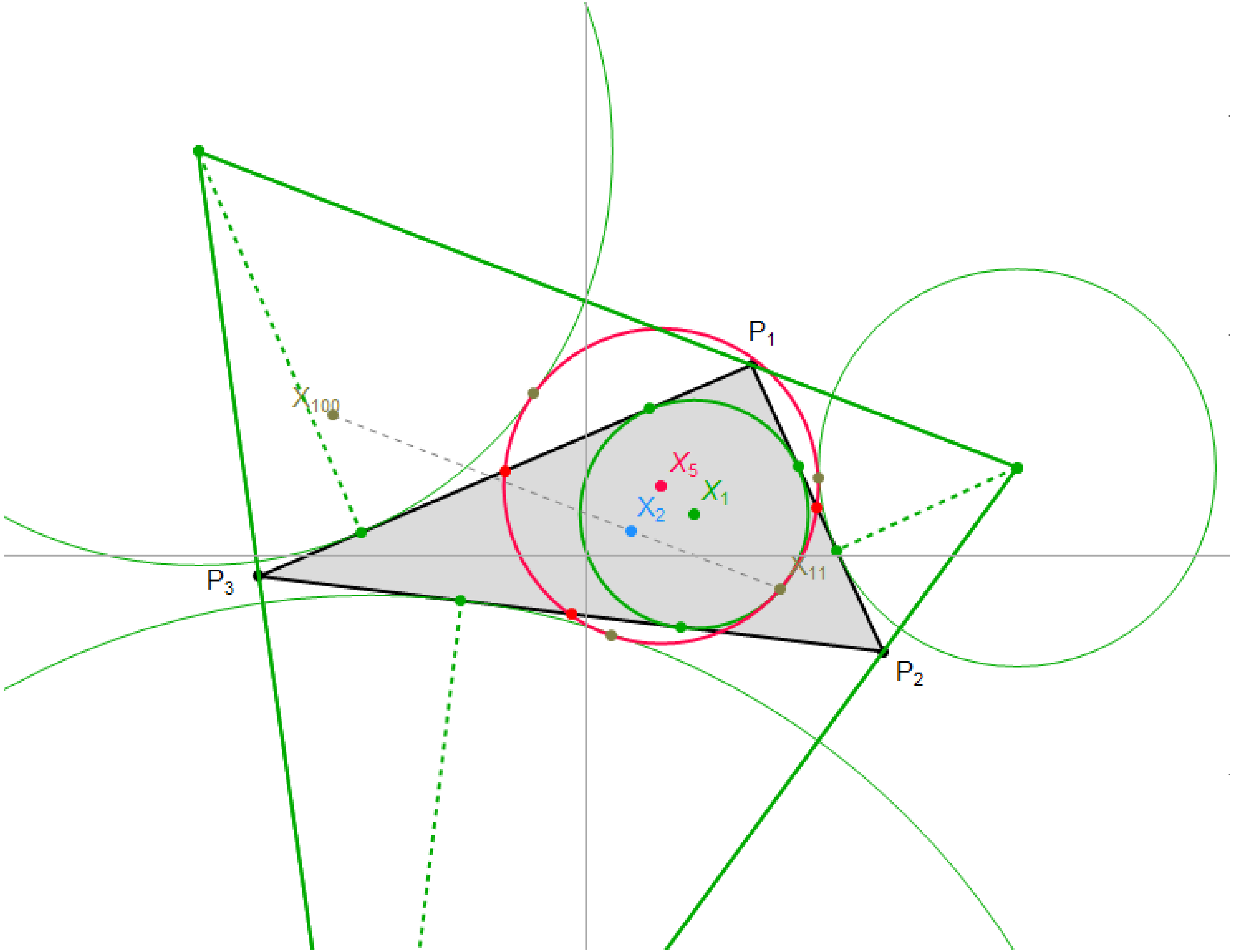
 

BRAZILIAN HOMAGE TO GASPARD! VIDEO



ELASTIC TRAJECTORY ELLIPTIC BILLIARD

APPLET

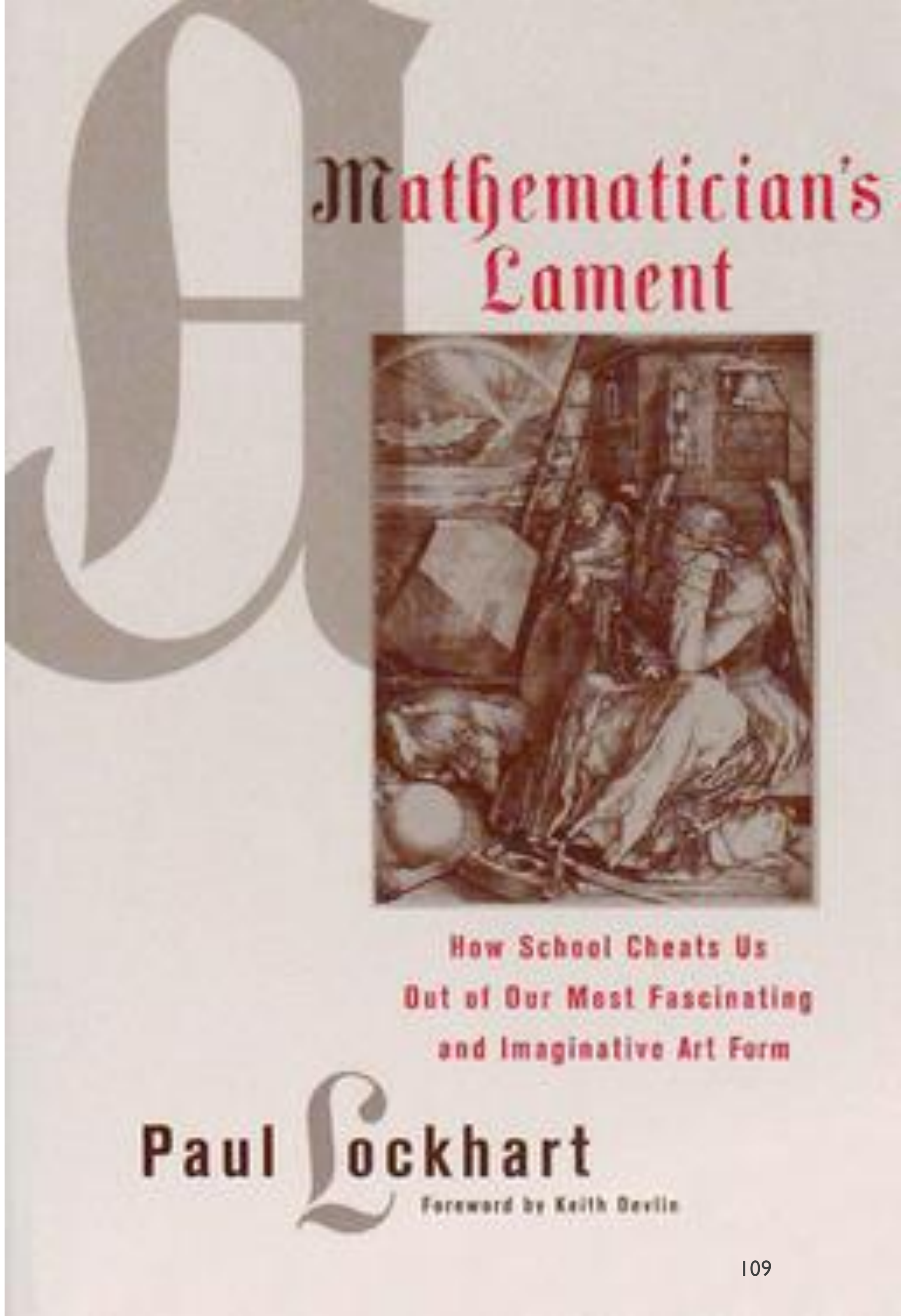


EXCENTRAL OBJECTS:

1) EXCENTRAL TRIANGLE
2) EXCIRCLES
3) EXTOUCHPOINTS

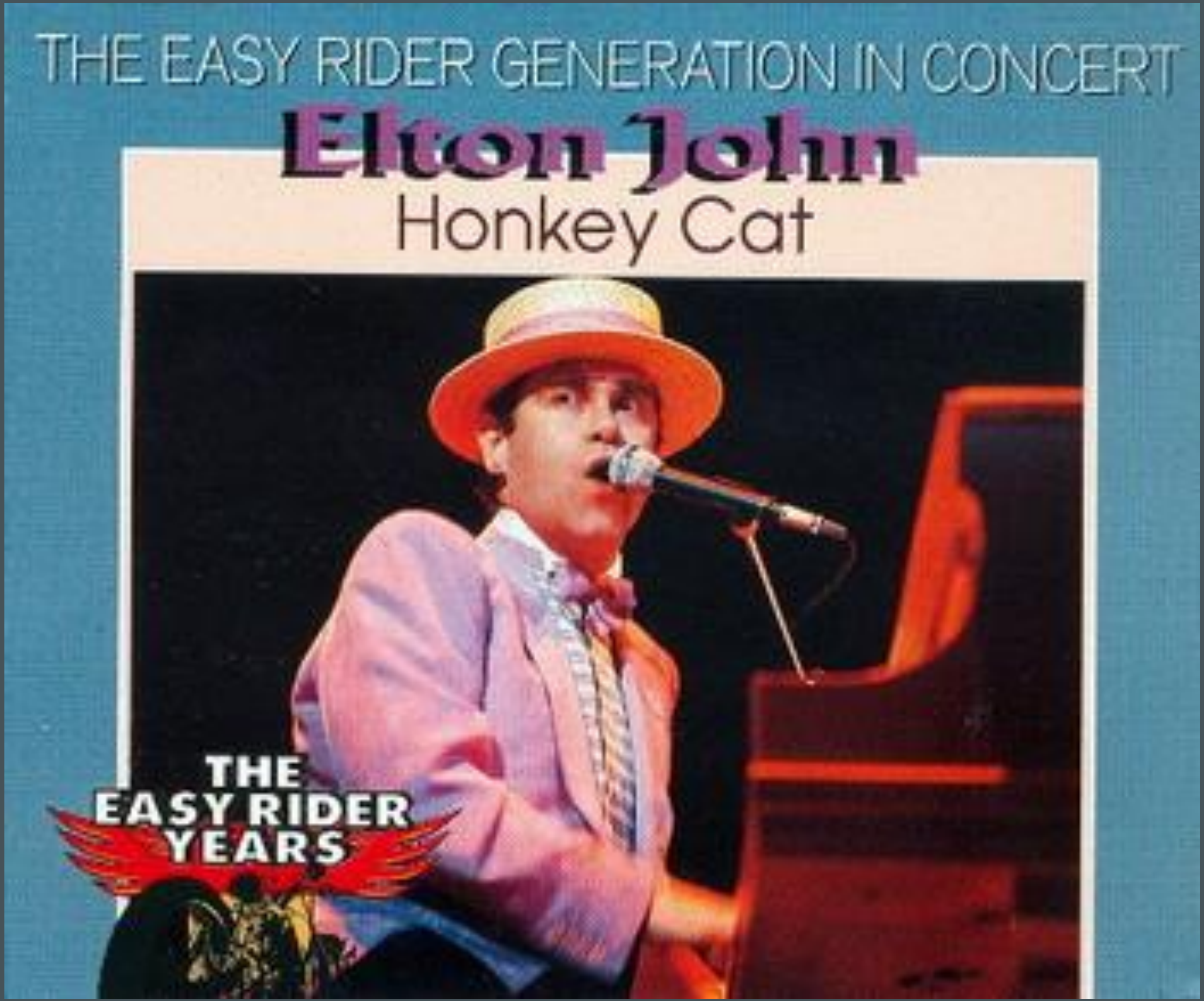
PAUL LOCKHART:

- Mathematics is about removing obstacles to our intuition and keeping simple things simple.
- **Math is not about following directions, it's about making new directions.**

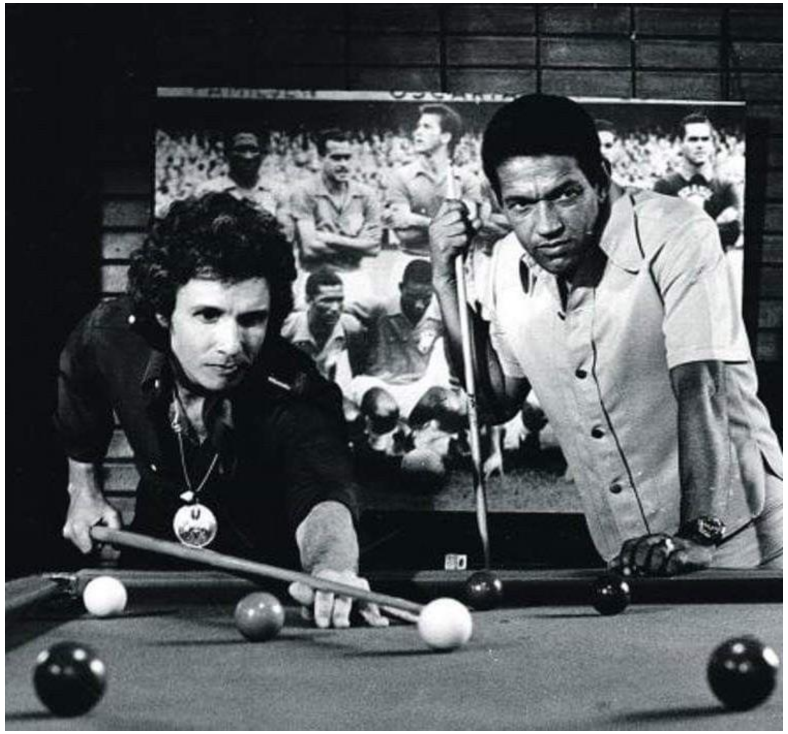


TRYING TO FIND GOLD IN A SILVER MINE

TRYING TO DRINK WHISKEY FROM A BOTTLE OF WINE

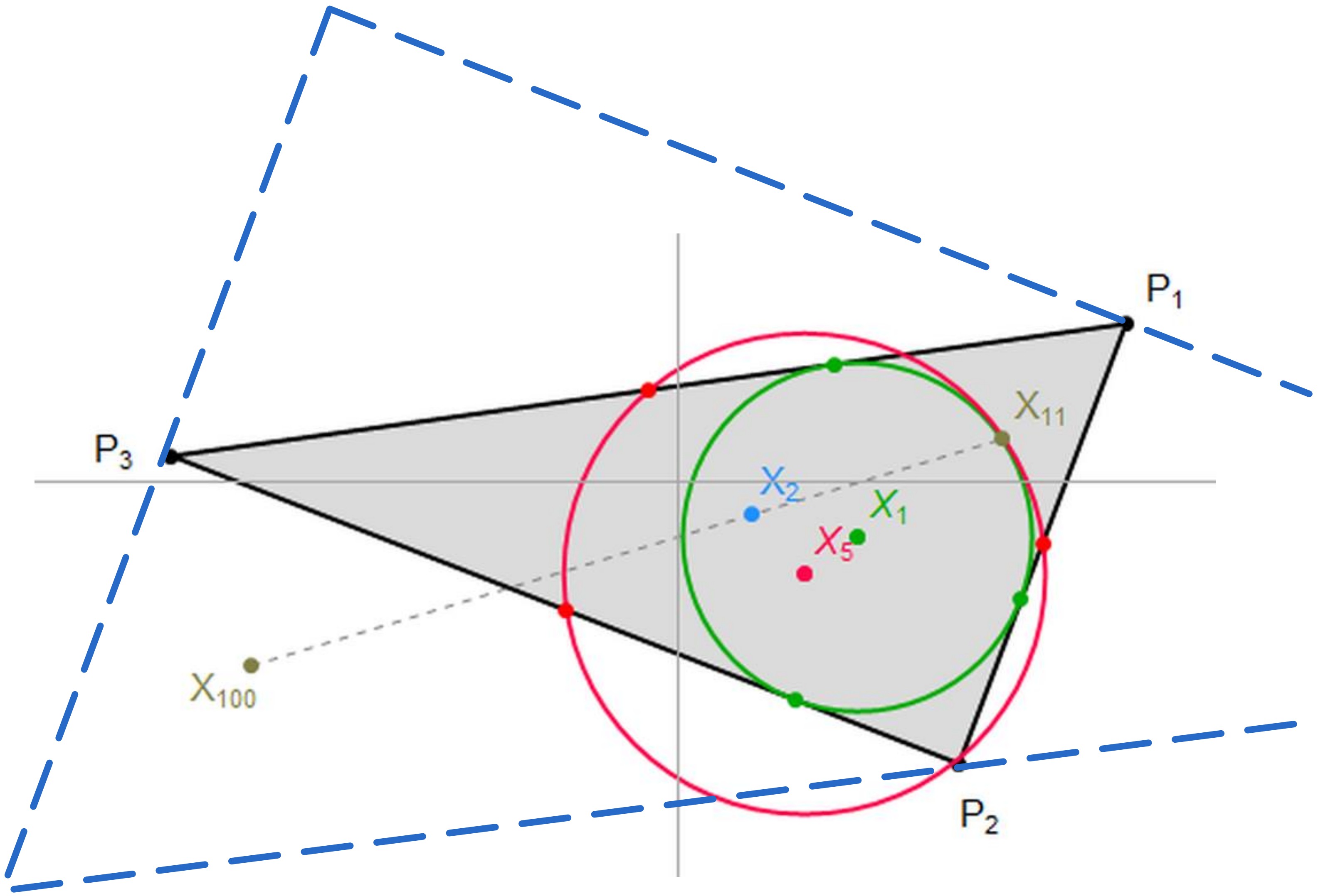


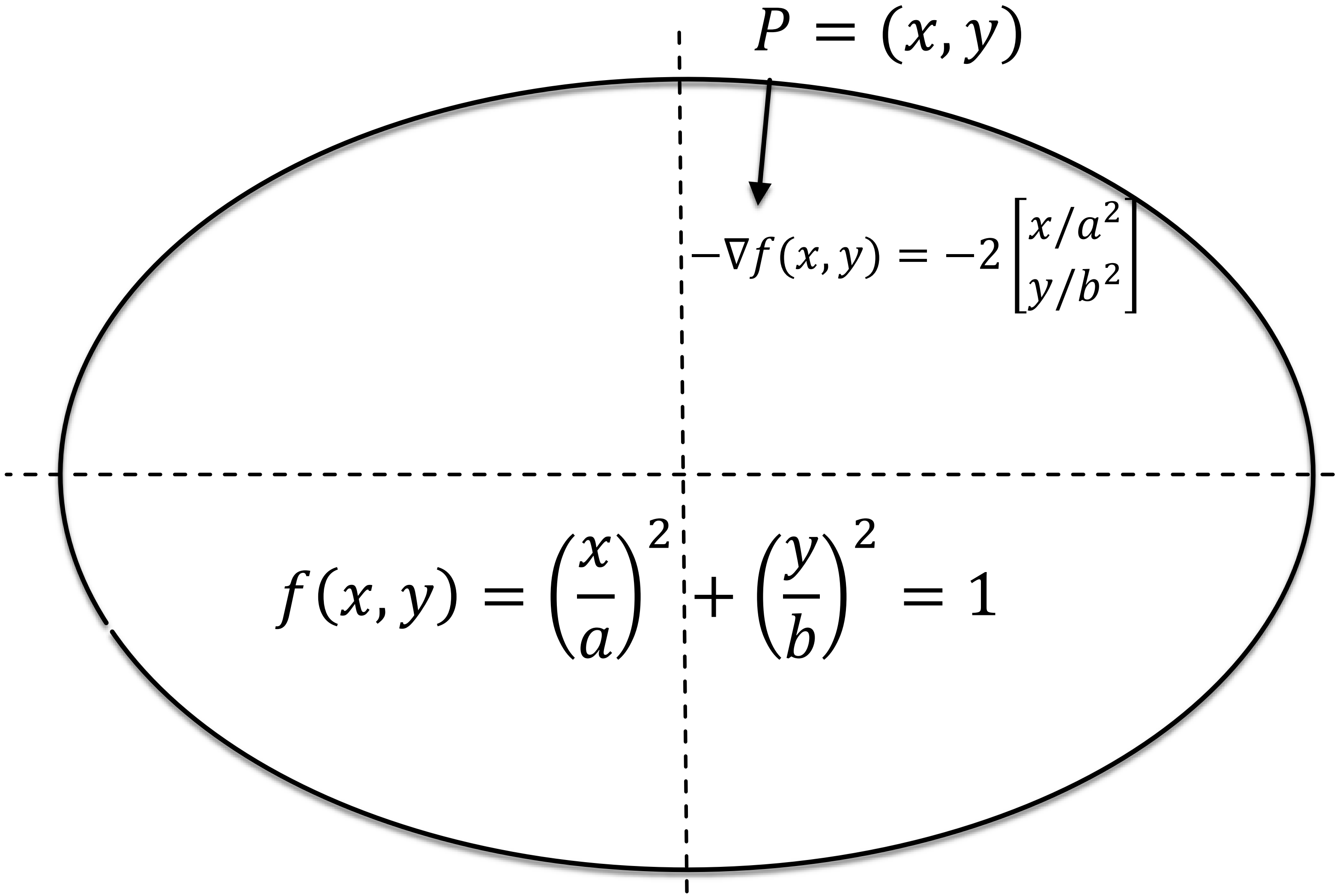
SPIROGRAPH

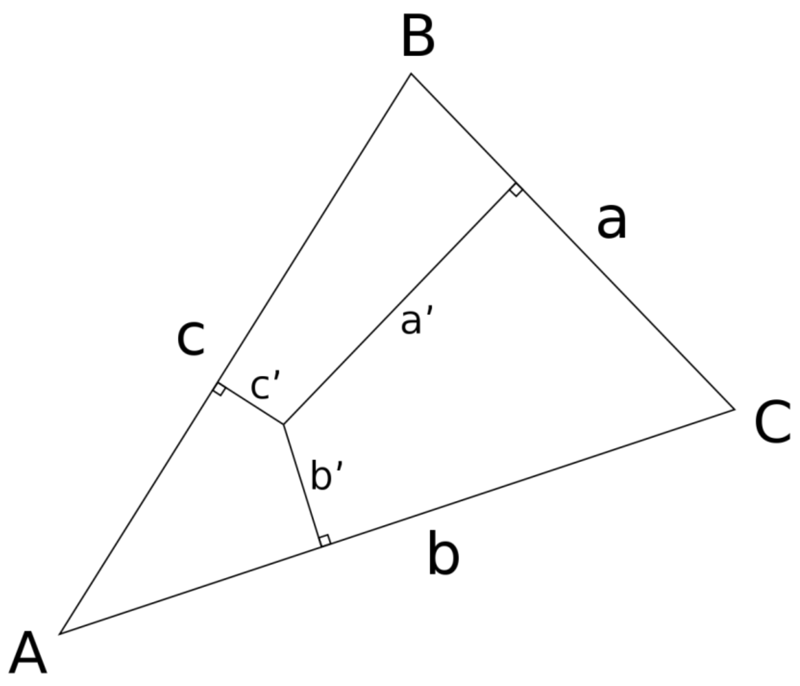


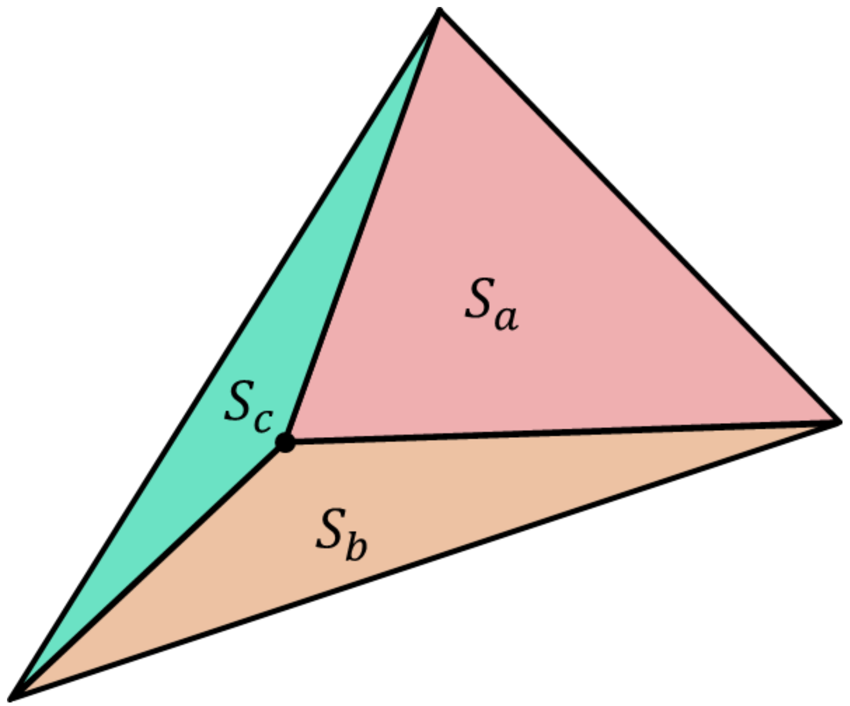
GARRINCHA E BILHARES:
JÁ COMBINOU COM OS RUSSOS?

X(100): ANTICOMPLEMENT OF FEUERBACH

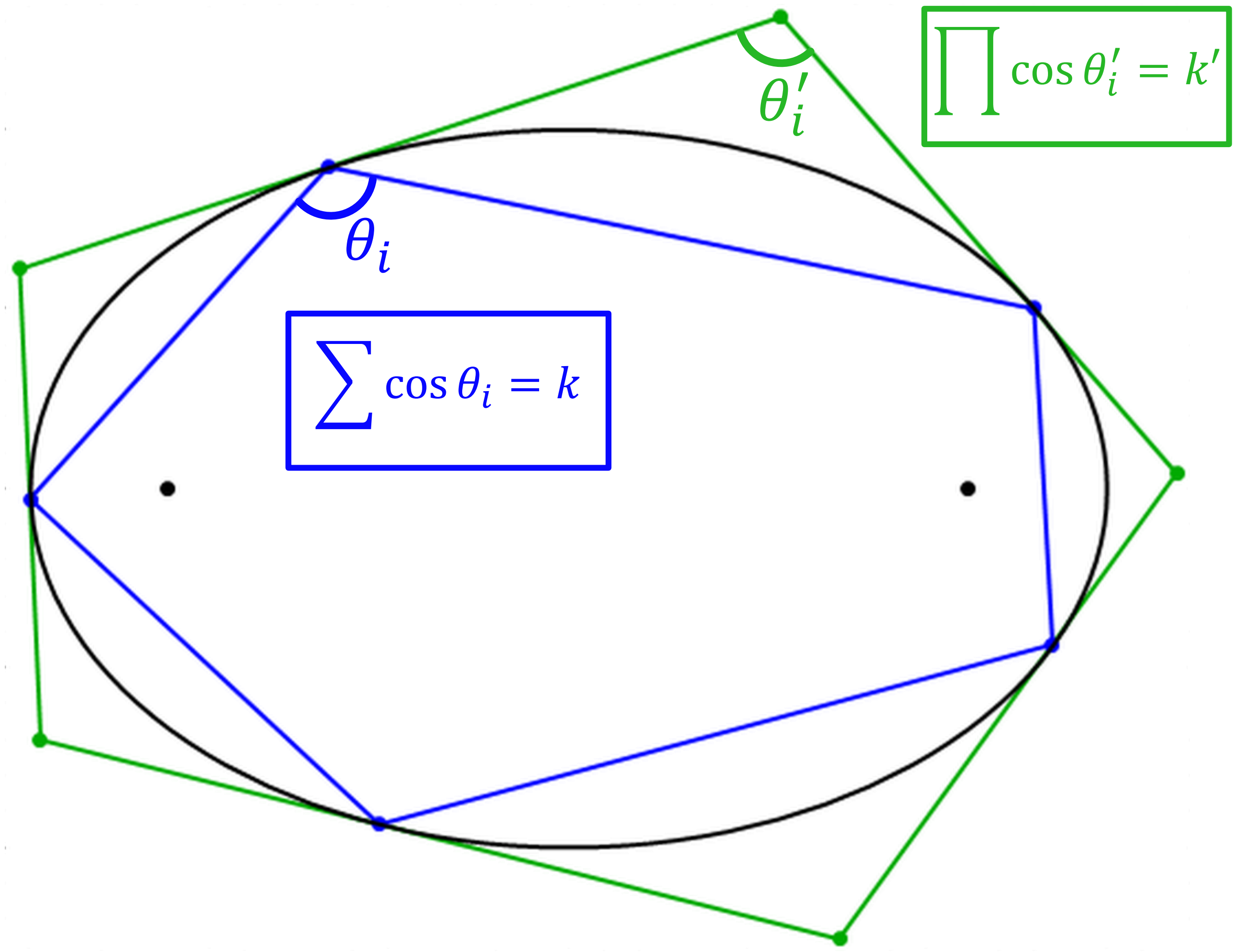



$P = (x, y)$
$-\nabla f(x,y) = -2\begin{bmatrix} x/a^2 \\ y/b^2 \end{bmatrix}$
$f(x,y) = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

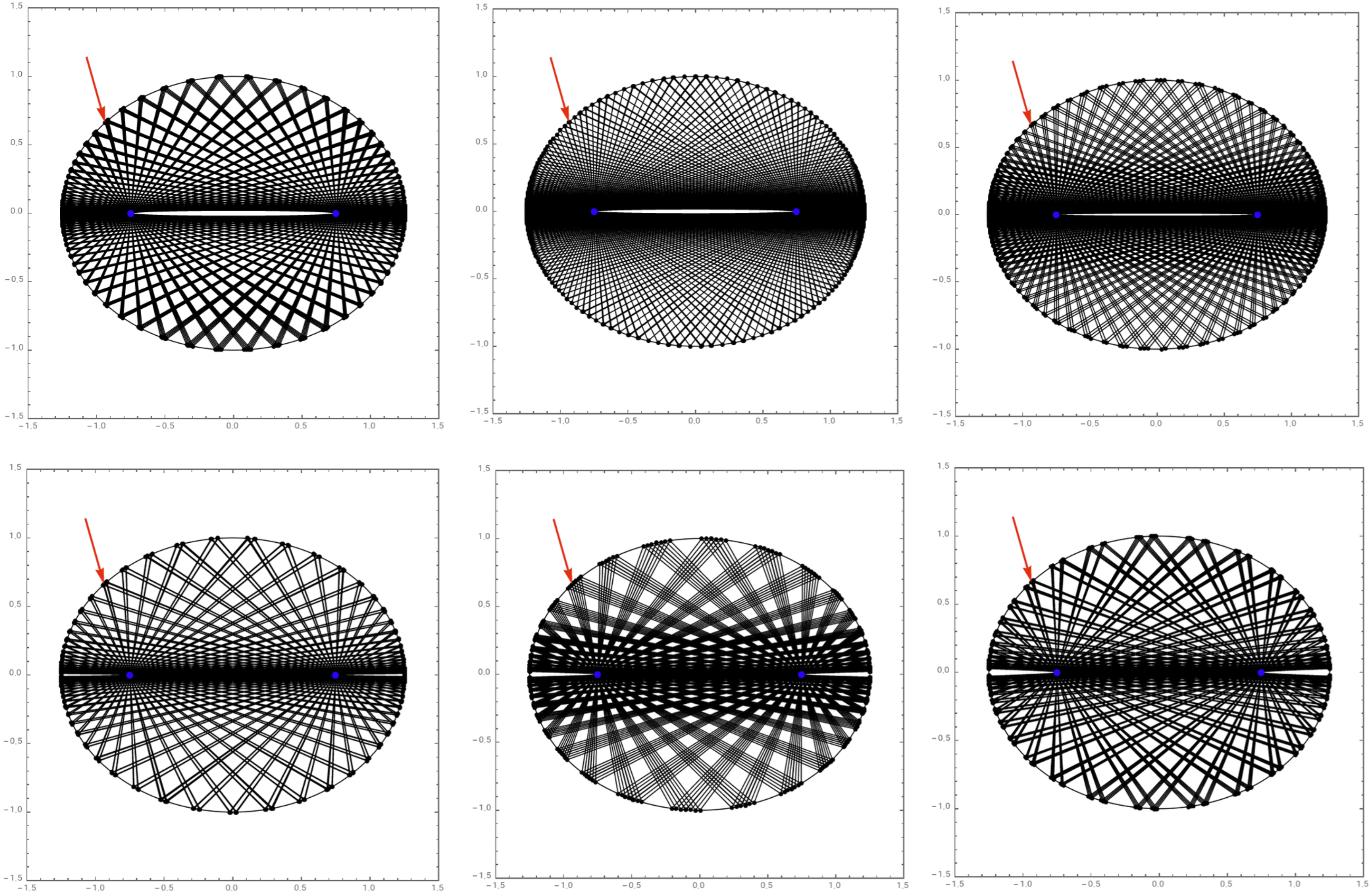




TRILINEAR VS BARYCENTRIC COORDINATES



$\prod \cos\theta_i' = k'$
θ_i'
θ_i
$\sum \cos\theta_i = k$


1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−1.5 −1.0 −0.5 0.0 0.5 1.0 1.5

A Property of Parallelograms Inscribed in Ellipses

Alain Connes and Don Zagier

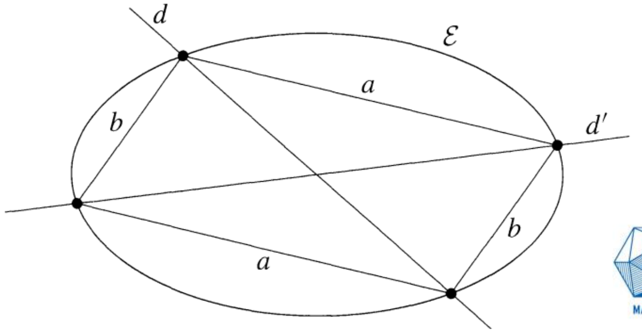
1. INTRODUCTION. The following surprising property of ellipses was observed by the physicist Jean-Marc Richard, in connection with a problem from ballistics [**2**, p. 843].

Theorem 1. *Let* $\mathcal{E}$ *be an ellipse and* $f(d, d')$ *the function of two diameters given by the perimeter of the parallelogram with vertices* $d \cap \mathcal{E}$ *and* $d' \cap \mathcal{E}$ *(Figure* 1*). Then*

$$f(d) := \sup_{d'} f(d, d')$$

is constant (independent of d*).*

In other words, the maximal perimeter of a parallelogram inscribed in a given ellipse can be realized by a parallelogram with one vertex at any prescribed point of the ellipse.



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