

# Elliptic Billiard: Surprising New Properties



D. Reznik – Upper West Soluções

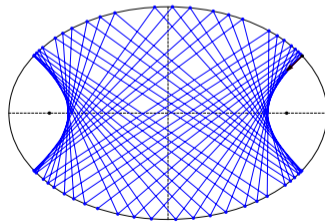
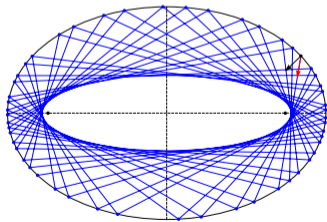
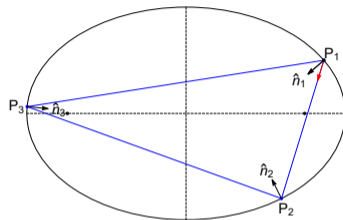
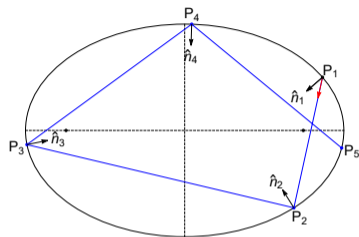
R. Garcia – Univ. Federal de Goiás

J. Koiller – Univ. Federal de Juiz de Fora

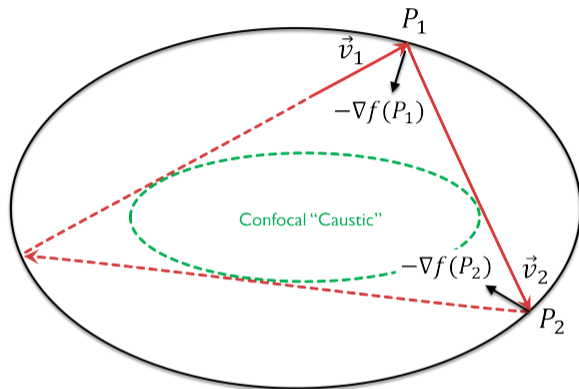
# Section 1

## Preliminaries

# Billiard Trajectories [video]



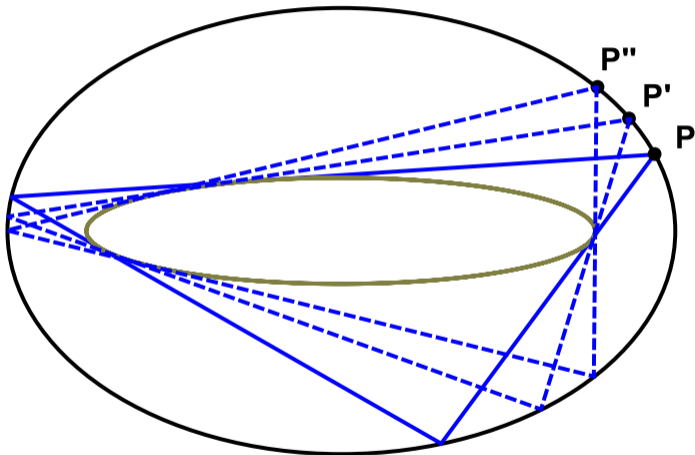
# Integrability & Conservation



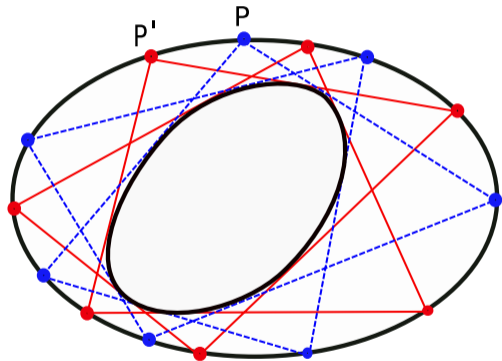
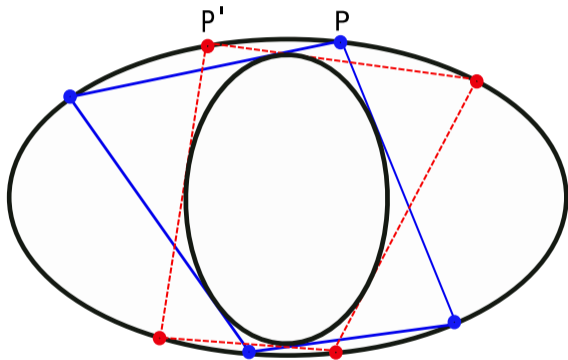
Energy  $\implies L = \text{const.}$

Joachimsthal's  $\implies \gamma = \frac{1}{2} \hat{v} \cdot \nabla f(P) = \text{const.}$

# $N = 3$ Caustic [video]



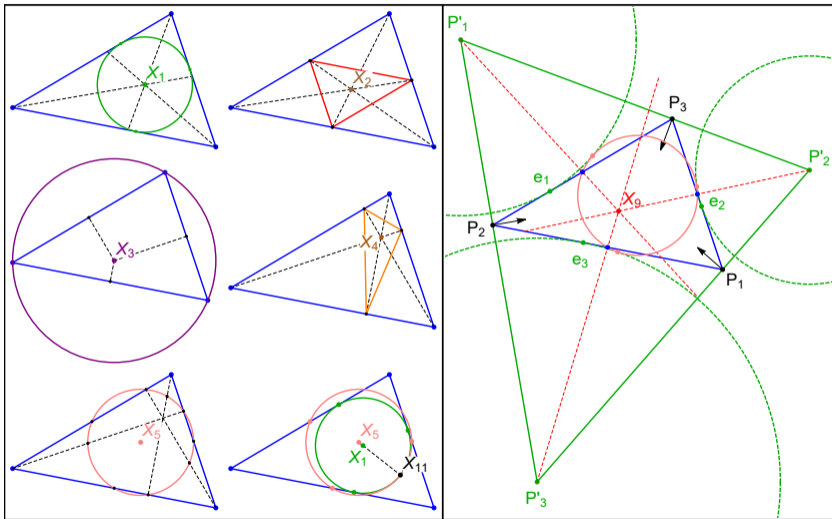
# Poncelet's Porism



## Section 2

# Locus Pocus

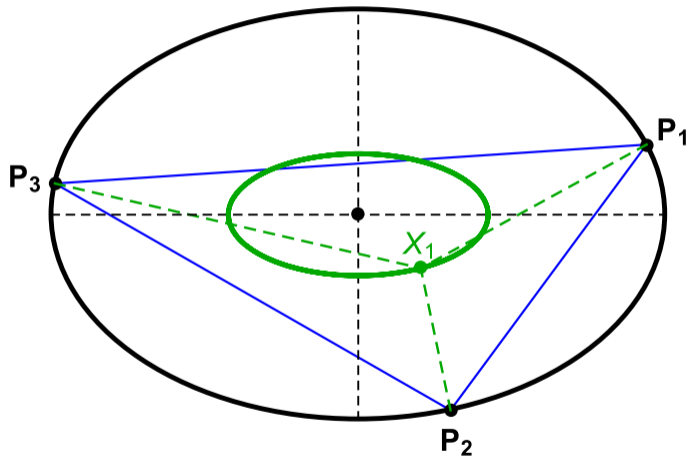
# Triangle Geometry



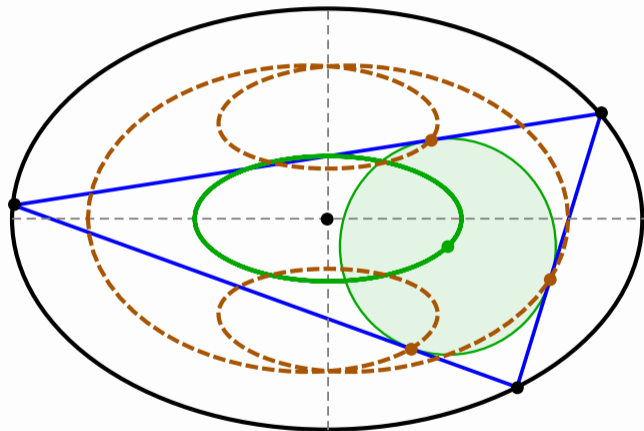
- $X_1$  Incenter
- $X_2$  Barycenter
- $X_3$  Circumcenter
- $X_4$  Orthocenter
- $X_5$  9-Point center
- $X_9$  Mittelpunkt
- $X_{11}$  Feuerbach Point



# Locus of Incenter is Elliptic [video]



# Locus Intouch Points is Sextic [video]



**Elliptic Billiard: Loci of Triangular Centers**

**Click: stop/go**  
**Drag: move billiard**  
**Wheel: zoom**

>>> **Billiard Aspect Ratio**  
 1.1  1.25  1.352  1.5  2.0  3.0

>>> **Elliptic Loci**

<input type="checkbox"/> X1*	<input type="checkbox"/> X2*	<input type="checkbox"/> X3*	<input type="checkbox"/> X4*
<input type="checkbox"/> X5*	<input type="checkbox"/> X7*	<input type="checkbox"/> X8	
<input type="checkbox"/> X10*	<input type="checkbox"/> X11*	<input type="checkbox"/> X12	
<input type="checkbox"/> X40*	<input type="checkbox"/> X88*	<input type="checkbox"/> X100*	
<input type="checkbox"/> X142*	<input type="checkbox"/> X144*	<input type="checkbox"/> X173	

>>> **Non-Elliptic Loci**

<input type="checkbox"/> X6~	<input type="checkbox"/> X15~	<input type="checkbox"/> X99~
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>>> **Notable Circles**

<input type="checkbox"/> Incircle	<input type="checkbox"/> Circum	<input type="checkbox"/> 9-Point
<input type="checkbox"/> Excircles	<input type="checkbox"/> AntIncire	<input type="checkbox"/> CosCirc

>>> **Derived Triangles**     Show Locus

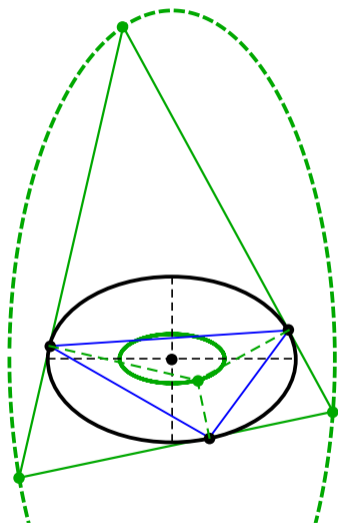
<input type="checkbox"/> $\Delta$ Exctr	<input type="checkbox"/> $\Delta$ Intouch	<input type="checkbox"/> $\Delta$ Extouch
<input type="checkbox"/> $\Delta$ Medial	<input type="checkbox"/> $\Delta$ Anticpl	<input type="checkbox"/> $\Delta$ Feuerb
<input type="checkbox"/> $\Delta$ Orthic	<input type="checkbox"/> OrthicInc	<input type="checkbox"/> $\Delta$ Euler
<input type="checkbox"/> $\Delta$ X15p	<input type="checkbox"/> $\Delta$ Symm	<input type="checkbox"/> $\Delta$ ExeSymm

>>> **Constructions**

<input type="checkbox"/> EulerLin	<input type="checkbox"/> Alt Feet	<input type="checkbox"/> X9Ins	<input type="checkbox"/> ExcPerps
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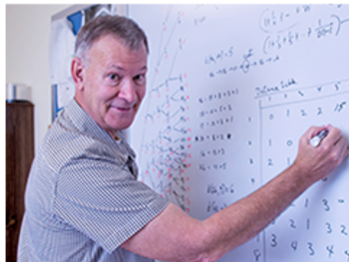
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# Locus of Excenters is Elliptic [video]



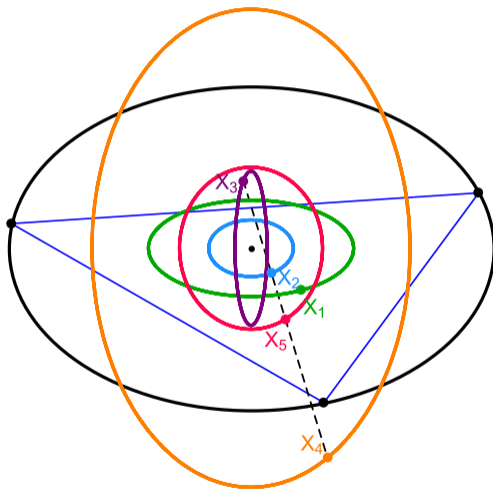


**This is PART 1: Introduction and Centers X(1) - X(1000)**

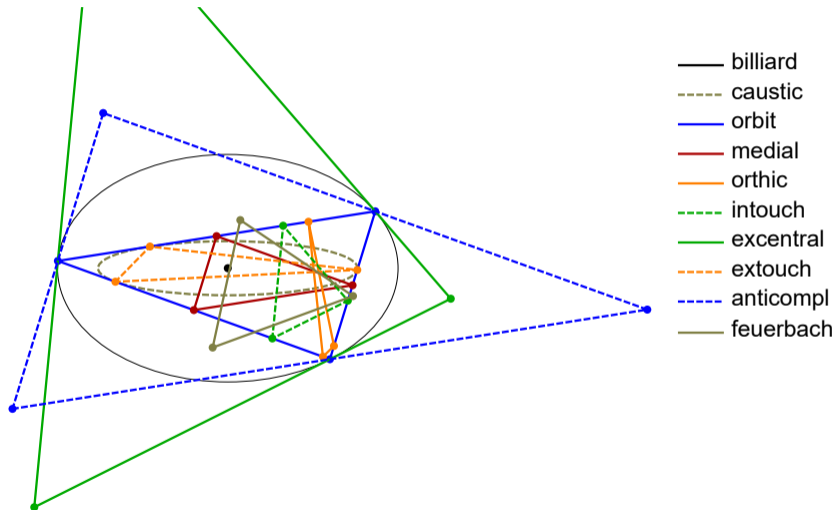


<a href="#">PART 1:</a>	Introduction and Centers X(1) - X(1000)
<a href="#">PART 2:</a>	Centers X(1001) - X(3000)
<a href="#">PART 3:</a>	Centers X(3001) - X(5000)
<a href="#">PART 4:</a>	Centers X(5001) - X(7000)
<a href="#">PART 5:</a>	Centers X(7001) - X(10000)
<a href="#">PART 6:</a>	Centers X(10001) - X(12000)
<a href="#">PART 7:</a>	Centers X(12001) - X(14000)
<a href="#">PART 8:</a>	Centers X(14001) - X(16000)
<a href="#">PART 9:</a>	Centers X(16001) - X(18000)
<a href="#">PART 10:</a>	Centers X(18001) - X(20000)
<a href="#">PART 11:</a>	Centers X(20001) - X(22000)
<a href="#">PART 12:</a>	Centers X(22001) - X(24000)
<a href="#">PART 13:</a>	Centers X(24001) - X(26000)
<a href="#">PART 14:</a>	Centers X(26001) - X(28000)
<a href="#">PART 15:</a>	Centers X(28001) - X(30000)
<a href="#">PART 16:</a>	Centers X(30001) - X(32000)
<a href="#">PART 17:</a>	Centers X(32001) - X(34000)
<a href="#">PART 18:</a>	Centers X(34001) - X(36000)
<a href="#">PART 19:</a>	Centers X(36001) - X(38000)
<a href="#">PART 20:</a>	Centers X(38001) - X(40000)

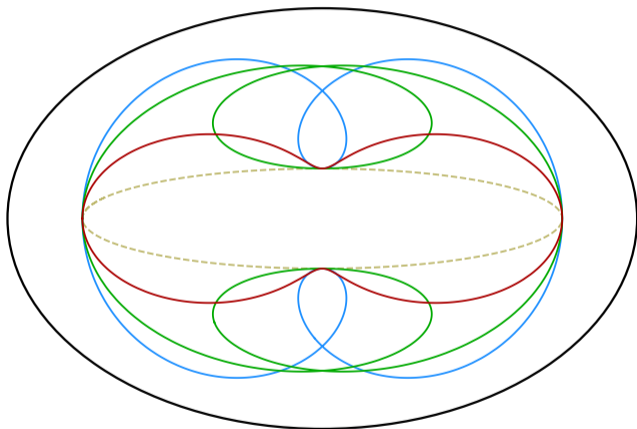
# Elliptic Loci of Major Triangular Centers [video]



# Derived Triangles [video]



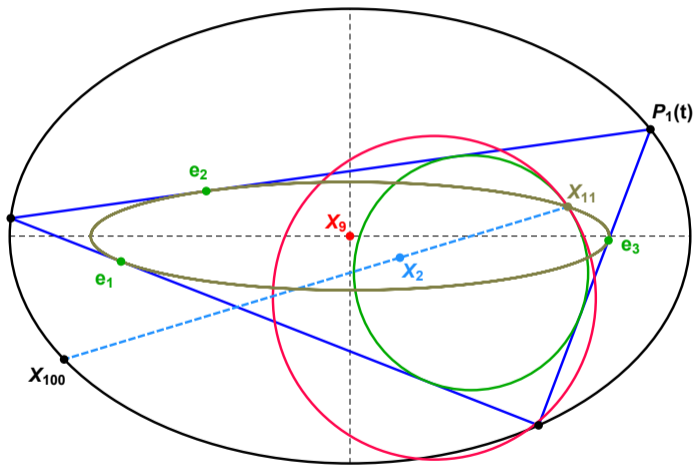
# Loci of Vertices of Derived Triangles [video]



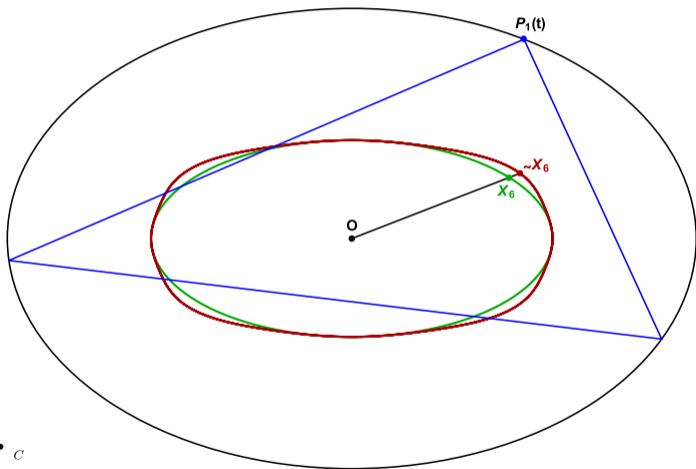
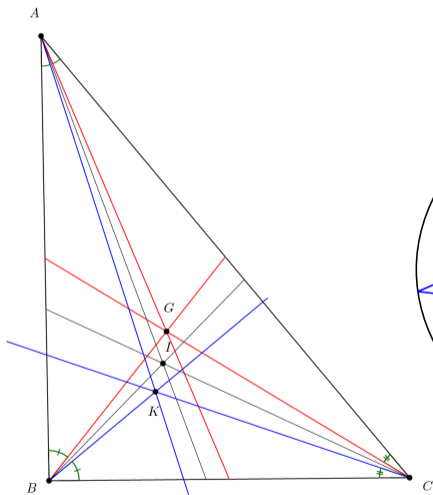
- Billiard
- $X_{11}$  & Extouch
- Intouch
- Feuerbach
- Medial



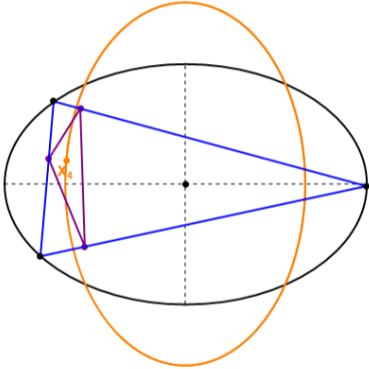
# Feuerbach, Anticomplement, and Extouchpoints [video]



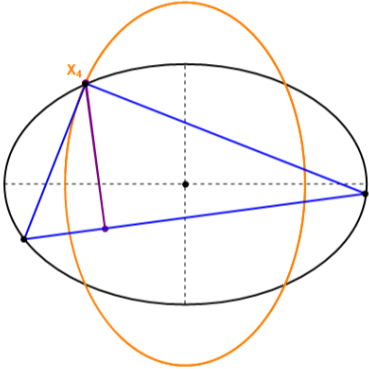
# Symmedian Point: Quasi-Elliptic Convex Quartic



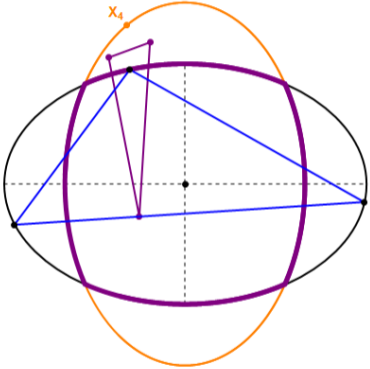
# Orthic Incenter: Four Kinks [video]



(a)

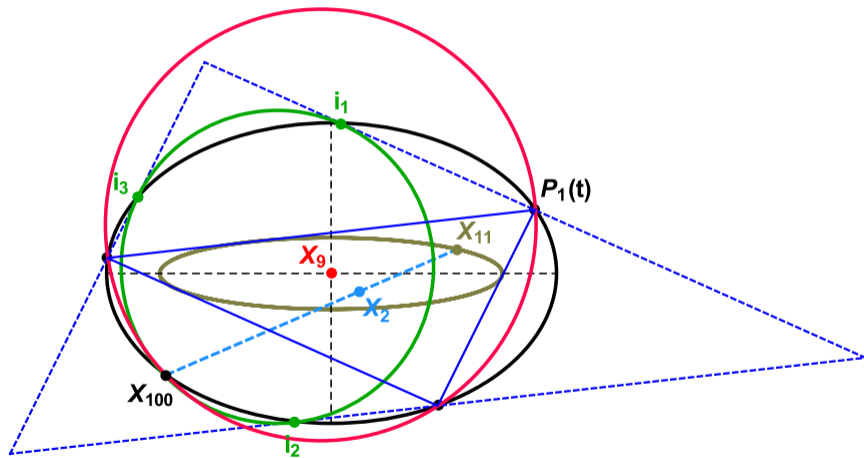


(b)

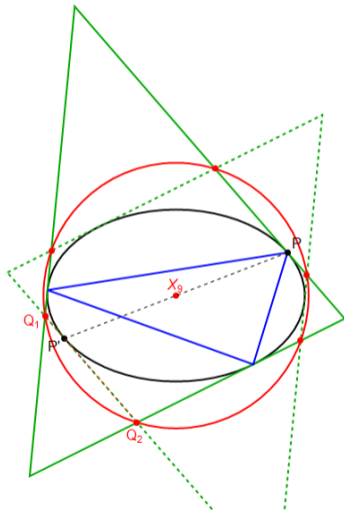


(c)

# Intouchpoints of Anticomplementary Track Billiard [video]



# A Circular Locus! [\[video\]](#)

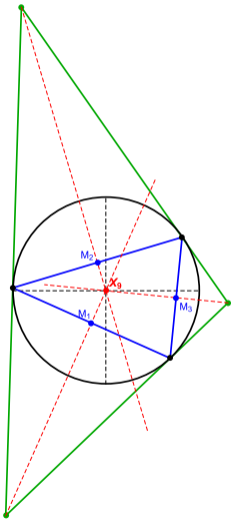
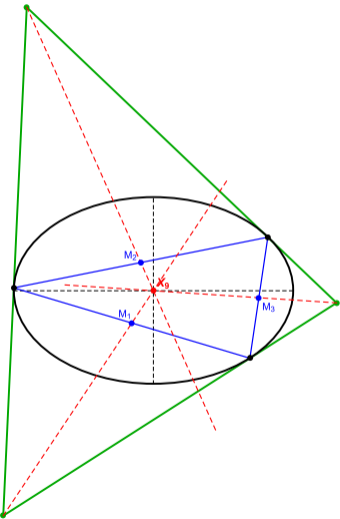


$$r^* = \frac{1}{\gamma}$$

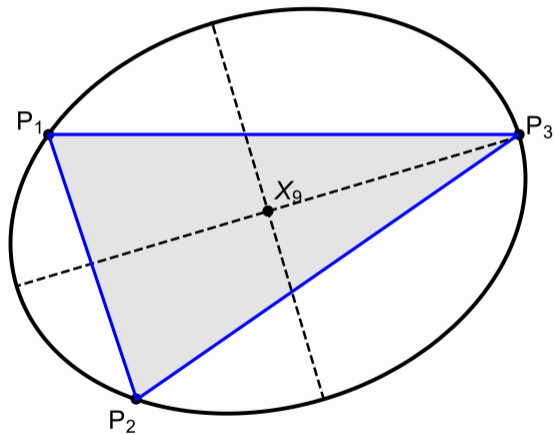
## Section 3

### *A Point Locus!*

# Mittelpunkt is Stationary [video]



# Circumbilliard [video]

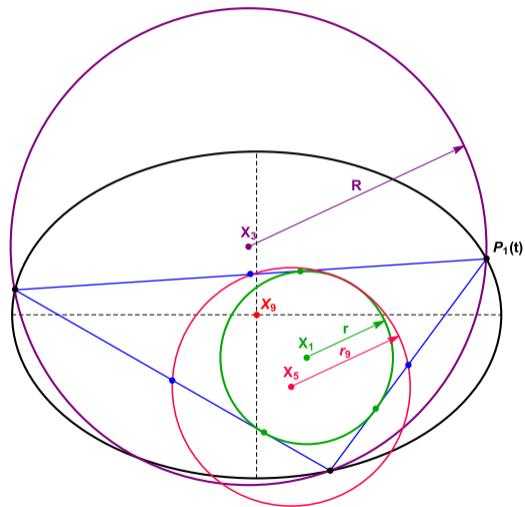




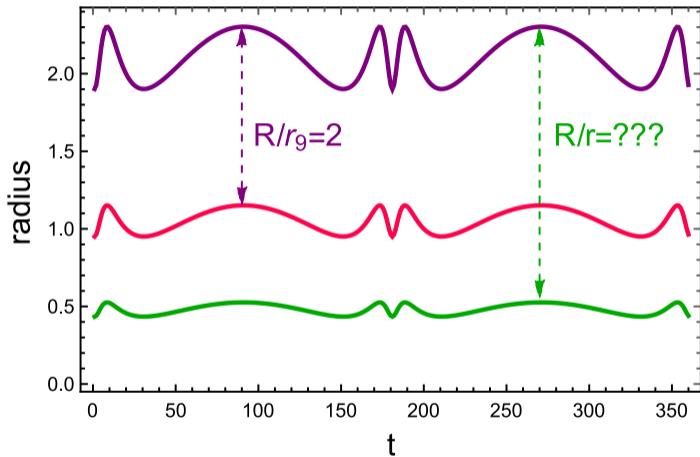
## Section 4

# The Beautiful Ratio

# Three Little Radii [video]

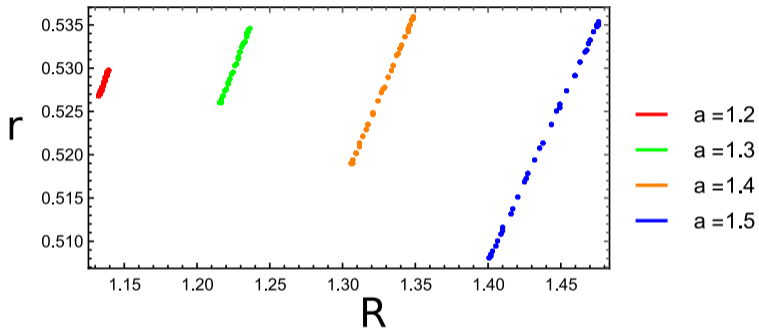


# Plotting Radii



- $r$  (inradius)
- $R$  (circumradius)
- $r_9$  (9pt-radius)

# A New Constant of Motion!



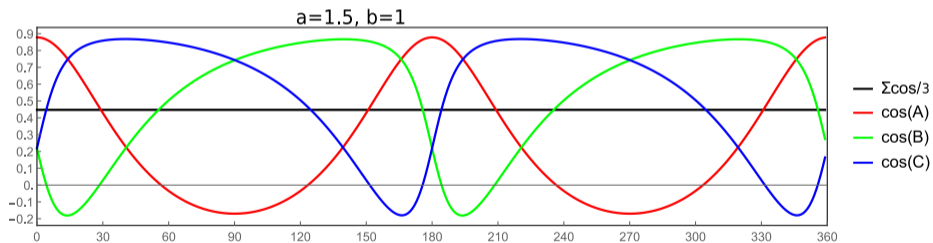
$$\frac{r}{R} = \gamma L - 4$$

## Section 5

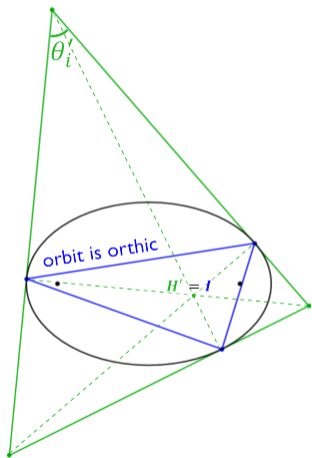
# Three Amazing Corollaries

# Sum of Orbit Cosines is Conserved!

$$\sum_{i=1}^3 \cos \theta_i = 1 + \frac{r}{R} = \gamma L - 3$$

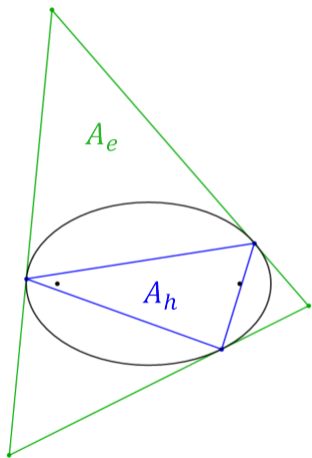


# Product of Excentral Cosines is Conserved! [\[video\]](#)



$$\prod_{i=1}^3 |\cos \theta'_i| = \frac{1}{4} \frac{r_h}{R_h} = \frac{\gamma L}{4} - 1$$

# Orbit-to-Excentral Area Ratio is Conserved!



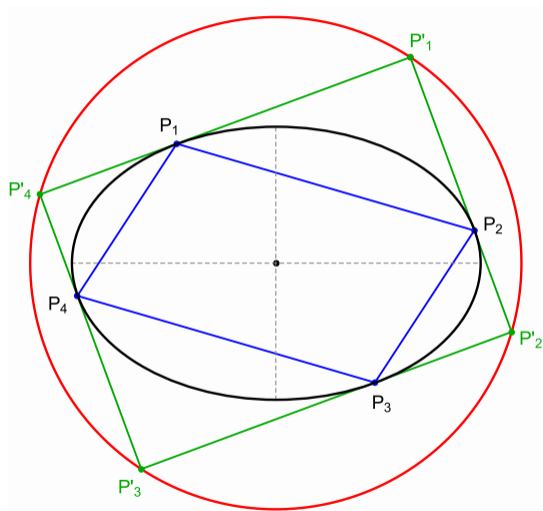
$$\frac{A_h}{A_e} = \frac{1}{2} \frac{r_h}{R_h}$$



## Section 6

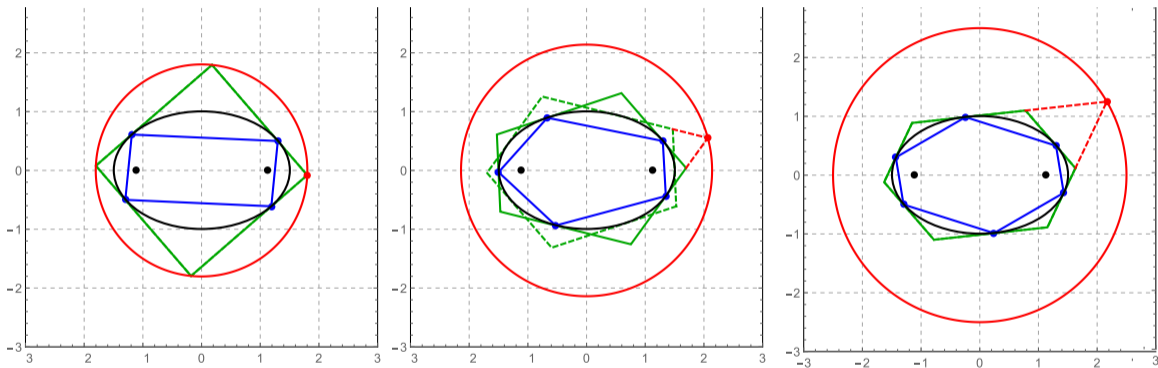
### Generalize $\forall N$

# Monge's Orthoptic Circle [video]



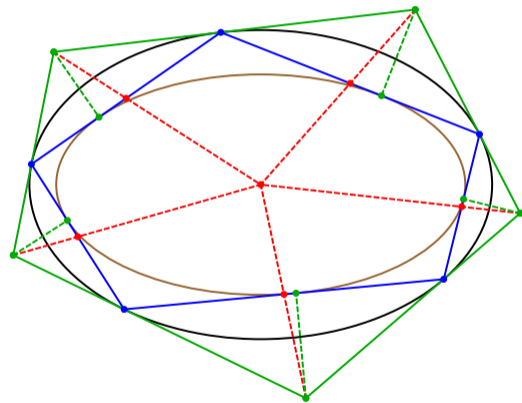
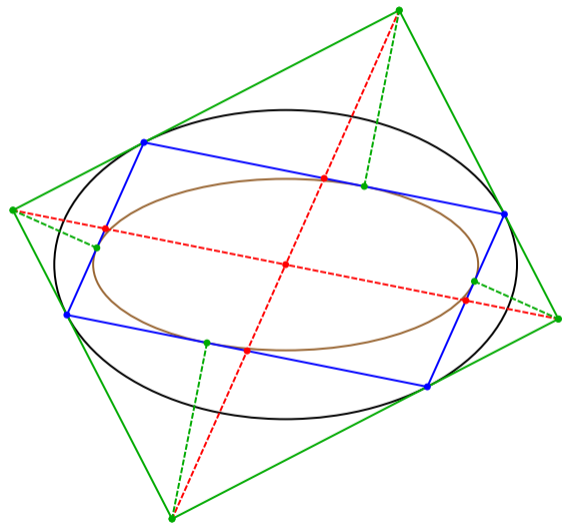
- \* Stationary Circle
- \* Rectangle diagonals.
- \*  $\sum \cos \theta_i = 0$ .
- \*  $\prod \cos \theta'_i = 0$ .

# Generalized Stationary Circle [video 1] and [video 2]

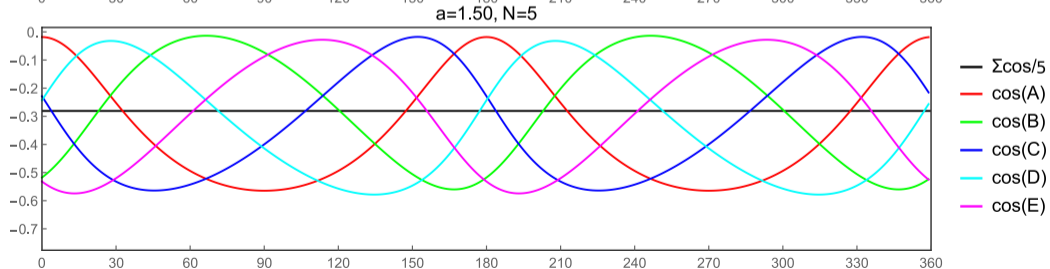
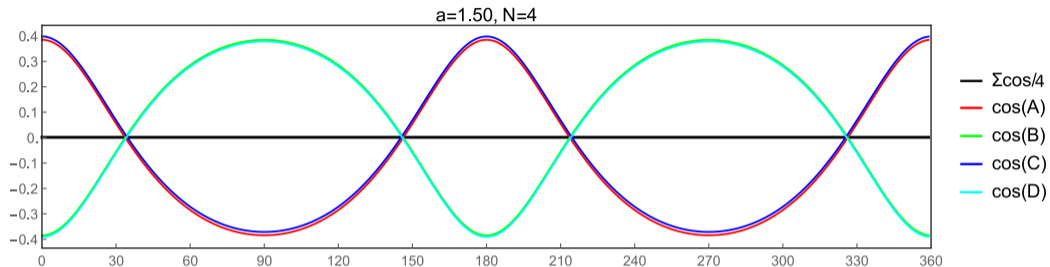


$\Rightarrow E'_i \cap -E'_{i+1}$  is a stationary circle of  $r^* = 1/\gamma$

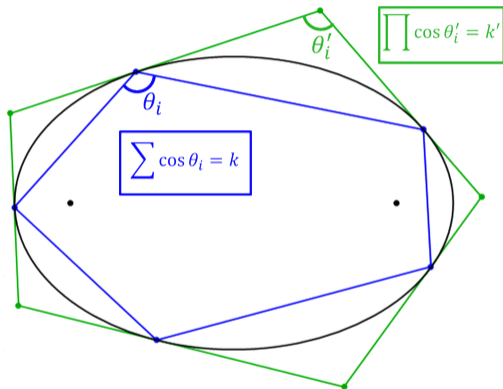
# $N > 3$ Mittenpunkt [video 1] and Extouchpoints [video 2]



# Cosine Sums for $N > 3$



# Sum, Product, Area Ratios are Conserved!



$\Rightarrow \sum_{i=1}^N \cos \theta_i$  conserved  $\forall N$ .

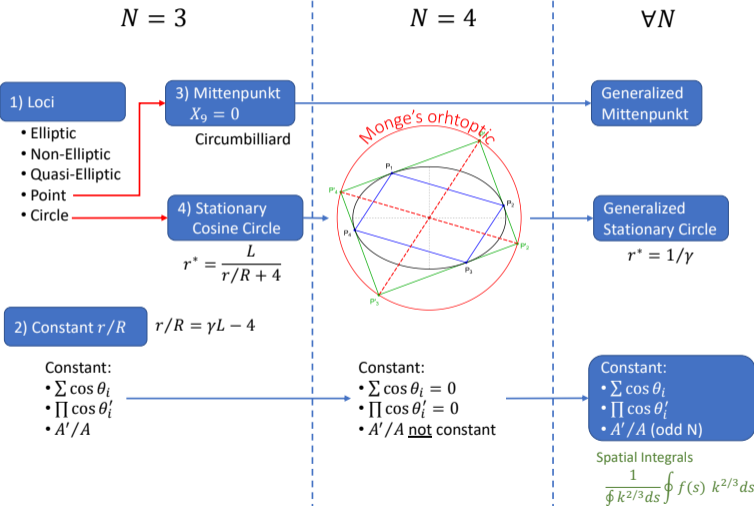
$\Rightarrow \prod_{i=1}^N \cos \theta'_i$  conserved  $\forall N$ .

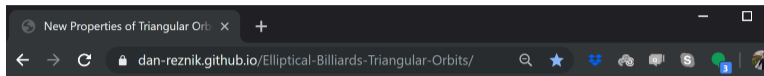
$\Rightarrow A'/A$  conserved  $\forall N$  **odd**.

# Summary

## 0) Integrability constants

- Perimeter  $L$
- Angular Mom.  $\gamma$





## 1 Introduction

2 Summary of Main Results

3 Results

4 Conclusion

5 Media by the Authors

6 Glossary of Terms

7 Appendix

References

# New Properties of Triangular Orbits in Elliptic Billiards

Dan Reznik, Ronaldo Garcia, Jair Koiller

Last update: 2019-09-04 14:42:15



## Assets and Downloads

- **Videos**, applets, images: [here](#)
- Interactive [applet](#).
- IMPA [talk](#) on July 29, 2019. You can **enable English subtitles** on YouTube.
- IMPA [slides](#).
- Mathematical Intelligencer [draft](#), in preparation.



# Questions

- \*  $N = 3$ : why is a locus is elliptic or non?
- \* Invariants for Ellipsoidal (3d) Billiard?
- \* Invariants in self-intersecting orbits? ( $N=4$  and  $N=5$ )
- \* Invariants in non-billiard (Poncelet) orbit families?
- \* Invariants for orbits on sphere?

**Thanks!**



dreznik[[@](mailto:dreznik@gmail.com)]gmail[.]com  
ragarcia[[@](mailto:ragarcia@ufg.br)]ufg[.]br  
jairkoiller[[@](mailto:jairkoiller@gmail.com)]gmail[.]com